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Photoemission from a Laser-Driven Electron Wave Packet

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ABSTRACT

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We use quantum electrodynamics (QED) to investigate the possibility of radiative interference from a single laser-driven electron wave packet. Intuition gleaned from classical electrodynamics suggests that radiation from a large electron wave packet might interfere destructively when different regions of the packet oscillate out of phase with each other. We show that when the incident light is represented with a multi-mode coherent state, the relative phases of the electron's constituent momenta have no influence of the amount of scattered light. Hence, the radiation does not depend on the amount of free-particle spreading experienced by the electron before the interaction. This result is shown to hold to all orders of perturbation theory. We extend our conclusions using the Furry picture of QED, where the (now-classical) incident light pulse is treated non-perturbatively with Volkov functions. We connect our results to a first-quantized picture by comparing transition probabilities between QED and semiclassical models. We are able to match these probabilities by choosing the classical scattered light field to be a single mode with energy $\hbar\omega'$.

Keywords: Quantum Electrodynamics, Theory, Photoemission, Interference

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Contents

Chapter 1

Introduction

1.1 Does Electron Wave-Packet Size Matter?

Classical radiation theory has been well understood for over a century. Maxwell's equations, although difficult to solve in practice, can be solved formally using Green functions and potentials. Even at the undergraduate level, one can gain good intuition for the radiation generated by *classical* charge densities and currents.

Maxwell's equations can be written in relativistic notation as

$$\frac{1}{c^2}\frac{\partial^2 A^\nu}{\partial t^2} - \nabla^2 A^\nu = \partial^\mu \partial_\mu A^\nu = \frac{4\pi}{c} J^\nu \tag{1.1}$$

in the Lorenz gauge, where $\partial_\mu A^\mu = 0$. In this thesis, we employ the Minkowski metric $g^{\mu\nu}$ characterized by $(+,-,-,-)$ such that $a^\mu b_\mu = a^0 b^0 - \vec{a}\cdot\vec{b}$. Gaussian units are primarily used in this chapter. The electric and magnetic fields may be derived from the potential $A^\nu(x)$ via

$$\begin{aligned} \vec{E}(x) &= -\nabla A^0(x) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(x) \\ \vec{B}(x) &= \nabla \times \vec{A}(x) \end{aligned} \tag{1.2}$$

When written in terms of potentials, it is clear that Maxwell's equations support the propagation of waves of velocity c.

The solution to (1.1) in free space may be written formally as [1]

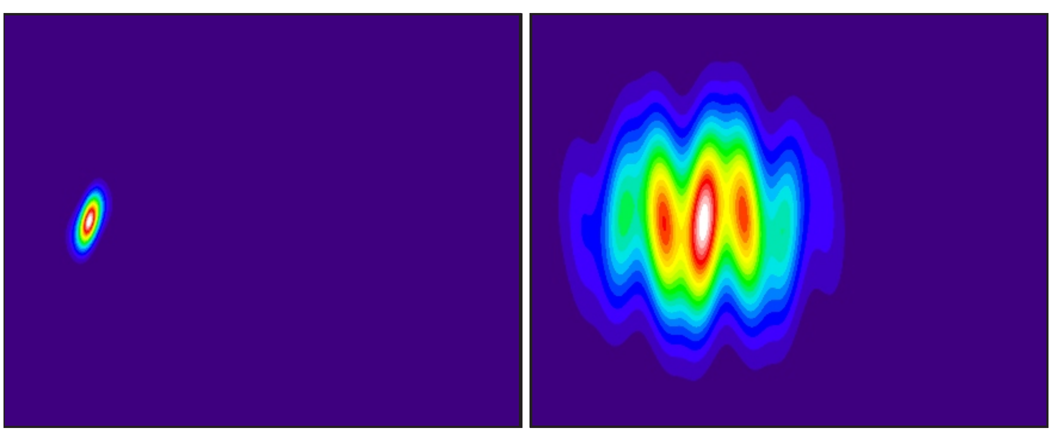
$$A^{\nu}(x) = \frac{4\pi}{c} \int d^4x' G_{ret}(x,x') J^{\nu}(x') \tag{1.3}$$

where $G_{ret}(x,x')$ is the retarded Green function for the wave equation, given by

$$G_{ret}\left(x,x'\right) = \frac{1}{2\pi}\theta(x_0 - x_0')\delta\left[\left(x - x'\right)^2\right] \tag{1.4}$$

where $(x-x')^2 \equiv (x-x')\cdot(x-x')$. The delta and Heaviside functions in (1.4) guarantee that the wave $A^{\nu}(x)$ is causally generated by the source $J^{\nu}(x')$. Clearly, the radiation from different regions of the current density $J^{\nu}(x_1)$ and $J^{\nu}(x_2)$ can interfere at space-time point x if $(x_1 - x)^2 = (x_2 - x)^2 = 0$. Interference also arises (via diffraction) if Maxwell's equations are solved in a finite region, in which case a boundary term must be added to (1.3). The prediction and measurement of interference (in, for example, Young's double slit experiment) are triumphs of the classical wave theory of light.

It is natural to wonder if this interference also arises in the quantum-mechanical problem where the source is a single (laser-driven) electron wave packet. Is it valid to use the electron's probability current (multiplied by electric charge) as the source term in (1.3)? If so, then one might



(a) Small packet (b) Large packet

Figure 1.1 A laser-driven electron wave packet that is (a) small compared to a wavelength, and (b) large compared to a wavelength.

expect interference effects to become salient when the size of the electron wave packet spans many wavelengths of the stimulating light, as shown in Fig. 1.1(b). The literature contains conflicting opinions on the matter [2–7].

There is some precedent for the notion that probability currents are analogous to classical charge currents. For instance, Schrodinger originally suggested that $e\psi^*(x)\psi(x)$ was a classical charge density when his wave equation was first published [8], and he believed this until his death in 1961 [9]. Other important figures of early quantum research, including Gordon [10] and Klein [11], appealed to this notion in their calculations. As we will show in Sec. 1.3, the gauge coupling of the QED Lagrangian also suggests that probability currents and charge currents are intrinsically related.

Even if one dismisses Schrodinger's interpretation and accepts the quantized radiation field, intuition gleaned from classical electrodynamics still suggests that radiation from a large electron wave packet might interfere, albeit probabilistically. Indeed, it would be tempting to regard the vector potential (1.3) as a probability amplitude; the intensity computed from such an amplitude would then characterize photon-detection probabilities and interference.

In this thesis, we investigate the possibility of radiative interference from a single electron using the fully-quantized framework of quantum electrodynamics (QED). We also comment on the implications for semiclassical models. Our starting premise in this analysis is that the present formulation of QED can answer this question correctly. As the truth of this premise is certainly up for debate, the Ware-Peatross research group at Brigham Young University is working to experimentally settle the issue.

1.2 Radiation as a First-Quantized Perturbation

Inasmuch as first-quantized quantum mechanics can describe much of atomic physics, we comment on the role of scattered radiation in the (first-quantized) Klein-Gordon equation. Although the relativistic quantum mechanical wave equations are plagued with interpretational difficulties [12], they might still give an intuition for radiation reaction. They moreover form the basis for semiclassical theories. We will see that the first-quantized framework fails unless a number of *ad hoc* modifications are imposed. In this thesis, the word "semiclassical" refers to theories that rely on the framework and perspective of (first-quantized) wave mechanics.

If the (spinless) electron has charge $e = -|e|$ and mass m and the electromagnetic potential is given by $A^\mu(x)$, the wave function must satisfy the Klein-Gordon equation:

$$\left[\left(i\hbar\partial^\mu - \frac{e}{c}A^\mu\right)^2 - m^2c^2\right]\psi = 0 \tag{1.5}$$

We may separate out the incident field A_i^μ from the scattered field A_s^μ via

$$A^\mu(x) = A_i^\mu(x) + A_s^\mu(x) \tag{1.6}$$

If the incident light field is a function of only $n\cdot x \equiv ct - \hat{n}\cdot\vec{x}$, then the Volkov states [13, 14]

$$\psi_{\vec{p}}^{v}(x) = \sqrt{\frac{mc^2}{VE_p}}\exp\left\{-i\frac{p\cdot x}{\hbar} + \frac{i}{\hbar n\cdot p}\int_{-\infty}^{n\cdot x}\left[ep\cdot A_i(\ell) - \frac{e^2}{2c}A_i(\ell)\cdot A_i(\ell)\right]d\ell\right\} \tag{1.7}$$

form a convenient basis for positive-energy solutions, solving the unperturbed Klein-Gordon equation

$$\left[\left(i\hbar\partial^\mu - \frac{e}{c}A_i^\mu\right)^2 - m^2c^2\right]\psi_{\vec{p}}^{v} = 0 \tag{1.8}$$

The orthonormality relation for Klein-Gordon Volkov states is

$$\frac{i\hbar}{2mc}\int d^3x\left[\psi_{\vec{p}}^{v*}\partial^0\psi_{\vec{p}'}^{v} - \psi_{\vec{p}'}^{v}\partial^0\psi_{\vec{p}}^{v*} - \frac{2e}{i\hbar c}A_i^0\psi_{\vec{p}}^{v*}\psi_{\vec{p}'}^{v}\right] = \delta_{\vec{p}\vec{p}'} \tag{1.9}$$

The unusual form of this "inner product" is dictated by the Klein-Gordon equation and the requirement that the "norm" of a state must be constant in time. One may verify that (1.9) is constant in time by comparing it to the conserved density of the Klein-Gordon equation [15].

We may ignore negative-energy states in the lowest order of perturbation theory because there are no intermediate states at this order, and it is assumed that the intensity is below the threshold of pair creation (where an intrinsically *single*-particle theory fails). Hence, the measured state of the system after interaction must be a single particle of positive energy. (At higher orders of perturbation theory, it is necessary to include negative-energy intermediate states [16].)

Designating $\frac{e}{c}A_s^{\mu}$ as a small perturbation, we find that the full problem is

$$\left[\left(i\hbar\partial^{\mu}-\frac{e}{c}A_i^{\mu}\right)^2-m^2c^2\right]\psi+\lambda V_{\text{int}}\psi+\lambda^2 V_{\text{int}}^{(2)}\psi=0 \tag{1.10}$$

where λ is the usual expansion parameter of perturbation theory, and we define the interaction terms as

$$\begin{aligned} V_{\text{int}} &\equiv -\frac{2ie}{c}A_s\cdot\partial+\frac{2e^2}{c^2}A_i\cdot A_s \\ V_{\text{int}}^{(2)} &\equiv \frac{e^2}{c^2}A_s\cdot A_s \end{aligned} \tag{1.11}$$

The solution to the perturbed problem may be expanded in the positive-energy Volkov basis via

$$\psi(x)=\sum_{\vec{p}}\left(\beta_{\vec{p}}^{(0)}+\lambda\beta_{\vec{p}}^{(1)}(t)+\dots\right)\psi_{\vec{p}}^{v}(x) \tag{1.12}$$

where we assume the initial condition $\beta_{\vec{p}}^{(n)}(-\infty)=0$ for $n\geq 1$. This means that $\left\{\beta_{\vec{p}}^{(0)}\right\}$ specifies the initial state of the electron. Appendix A derives an expression for the first-order transition amplitude:

$$\beta_{\vec{p}'}^{(1)}(\infty)=\frac{i}{2\hbar mc}\sum_{\vec{p}}\beta_{\vec{p}}^{(0)}\int d^4x\,\psi_{\vec{p}'}^{v*}V_{\text{int}}\psi_{\vec{p}}^{v} \tag{1.13}$$

Given a specific incident pulse A_i^{μ}, the computation of (1.13) still requires that one specify the scattered field A_s^{μ}. Classical electrodynamics holds that the electron must (one way or another) be the source of this field; however, *quantum mechanics* does not prescribe the form of A_s^{μ} at this level of the theory. With A_s^{μ} unspecified, (1.13) does not seem particularly useful. It turns out that quantum electrodynamics can salvage this first-quantized amplitude, as will be shown in Sec. 2.4. We now turn our attention to QED.

1.3 Gauge Invariance: the Launching Point of QED

Local gauge invariance plays a fundamental role in quantum electrodynamics because it leads directly to light-matter coupling. It is well known that the potential formulation of classical electrodynamics is not unique, as the local gauge transformation

$$A^{\mu} \to A^{\mu} + \partial^{\mu}\chi \tag{1.14}$$

does not make any measurable changes [1]. In classical electrodynamics, this is the meaning of "local gauge invariance." The transformation is said to be "local" because χ is a function of the space-time point x.

First-quantized wave mechanics also reflects local gauge invariance. One may introduce a light field $A^{\mu}(x)$ to the quantum wave equations as an alteration of the kinetic momentum operator:

$$p^{\mu}_{\text{kinetic}} = p^{\mu}_{\text{canonical}} - \frac{q}{c}A^{\mu} \tag{1.15}$$

This was the coupling introduced in the Klein-Gordon equation (1.5). The Dirac equation becomes

$$\left(i\hbar\gamma\cdot\partial - \frac{q}{c}\gamma\cdot A - mc\right)\psi = 0 \tag{1.16}$$

where γ^{μ} is a 4×4 Dirac matrix. If the potential is transformed according to (1.14), then one must correspondingly transform the wave function ψ via

$$\psi \to \exp\left[-\frac{iq\chi}{\hbar c}\right]\psi \tag{1.17}$$

so that the wave equation remains invariant. By inspection, one can see that the combined transformations (1.14) and (1.17) do not change the form of the Dirac, Klein-Gordon, and Schrodinger equations. Although the explicit form of the wave function and kinetic momentum operator depend on the gauge choice, there is no problem with the theory because a specific gauge choice does not affect measurable quantities [17]. That is, the theory remains gauge invariant.

In quantum electrodynamics, the principle of gauge invariance couples light and matter at the level of Lagrangian densities. We will proceed explicitly with the Dirac Lagrangian density, as it is more compact than its Klein-Gordon counterpart and will be the basis for most of our analysis. The Dirac equation for a free particle can be derived (*à* la Euler-Lagrange) from the Lagrangian density [12, 18]

$$\mathscr{L}_{\text{Dirac}}(x) = \bar{\psi}\left(i\hbar\gamma\cdot\partial - mc\right)\psi \tag{1.18}$$

where $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$. In the presence of an external field, this Lagrangian density becomes

$$\mathscr{L}_{\text{Dressed}}(x) = \bar{\psi}\left(i\hbar\gamma\cdot\partial - \frac{e}{c}\gamma\cdot A - mc\right)\psi \tag{1.19}$$

If A^{μ} is also a dynamical variable, then we must include the Lagrangian density for the free electromagnetic field [1]:

$$\mathscr{L}_{\text{EM}}(x) = -\frac{1}{16\pi}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right)\left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\right) \tag{1.20}$$

The full Lagrangian density (valid for QED) is given by the gauge-invariant quantity

$$\begin{aligned}\mathscr{L}_{\text{QED}} &= \mathscr{L}_{\text{Dressed}} + \mathscr{L}_{\text{EM}} \\ &= \bar{\psi}\left(i\hbar\gamma\cdot\partial - mc\right)\psi - e\bar{\psi}\gamma_{\mu}\psi A^{\mu} - \frac{1}{16\pi}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right)\left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\right)\end{aligned} \tag{1.21}$$

A similar Lagrangian density exists for Scalar Quantum Electrodynamics [19]:

$$\begin{aligned}\mathscr{L}_{\text{SQED}} = c^{2}\left(\partial_{\mu}\phi\right)\left(\partial^{\mu}\phi\right) - \frac{m^{2}c^{4}}{\hbar^{2}}\phi^{*}\phi - \frac{1}{16\pi}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right)\left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\right) \\ - \frac{iec}{\hbar}\left[\phi^{*}\left(\partial_{\mu}\phi\right) - \left(\partial_{\mu}\phi^{*}\right)\phi\right]A^{\mu} + \frac{e^{2}}{\hbar^{2}}A_{\mu}A^{\mu}\phi^{*}\phi\end{aligned} \tag{1.22}$$

where we now denote the Klein-Gordon field by ϕ.

Curiously enough, the equation of motion for the field A^{μ}, as derived from either (1.21) or (1.22), becomes

$$\partial_{\nu}\partial^{\nu}A^{\mu} = \frac{4\pi}{c}e\,j^{\mu} \tag{1.23}$$

in the Lorenz gauge, where j^μ is the expression for the Dirac or Klein-Gordon probability current. So far, the principle of gauge invariance appears to suggest that probability currents are on equal footing with classical charge currents. If this is indeed the correct interpretation, then large electron wave packets exhibit radiative interference (in the sense described in Sec. 1.1) and the scattered field A_s^μ is determined by (1.3). We will show that this interpretation and result turn out to be *incorrect* in quantum electrodynamics.

The procedure of second quantization described in the next section fundamentally changes how we interpret the couplings (1.21) and (1.22), and it supersedes the semiclassical interpretation given above. In quantum electrodynamics, $\psi(x)$ is *not* a probability amplitude and $A^\mu(x)$ is *not* the familiar vector potential. These objects become quantized field operators, thus making them distinct from the actual state of the system. In other words, the Klein-Gordon and Dirac wave equations are satisfied by abstract operators, not probability wave functions. We will see that this changes the physics embodied in (1.21) and (1.22), allowing the theory to include new and necessary features.

We note that the principle of gauge coupling can be generalized to transformations more complicated than (1.14) and (1.17). Other sets of such transformations form the basis of quantum chromodynamics and electroweak theory [20].

1.4 Quantum Electrodynamics

The QED Lagrangian density (1.21) is only the beginning of the quantum theory of radiation. The full theory must account for the existence of light quanta, the emission/absorption of these quanta, radiation reaction, the entanglement of electron-photon systems, etc. This section is presented as a big-picture overview of QED.

The framework of quantum electrodynamics is very similar to that of ordinary quantum mechanics. Several key features remain entirely intact:

- The set of quantum states forms a Hilbert space. We denote a generic vector with a 'ket' $|\psi\rangle$. These vectors are not functions of position $\vec{x}$; rather, they are abstract objects that "live" in the Hilbert space. One may produce a scalar by taking the inner product of a ket with a 'bra': $\langle\phi|\psi\rangle$.

- For every observable, there exists a corresponding linear Hermitian operator that acts on the vector space.

- Probability amplitudes are found by projecting a state $|\psi\rangle$ onto an observable's eigenvector $|n\rangle$ (where the eigenvalue of this generic eigenvector is n). The magnitude-squared of the projection, $|\langle n|\psi\rangle|^2$, represents the probability (or probability density) of measuring the corresponding eigenvalue n. The first-quantized wave function $\psi(x) \equiv \langle x|\psi\rangle$ is an example of such a projection. We account for degeneracy by introducing a complete set of commuting observables.

- The state of a system evolves (in the assumed Schrodinger picture) according to the Schrodinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.24}$$

  where H is a suitably-defined Hamiltonian operator.

- There exists a unitary operator $U(t,t_0)$, generated by Hamiltonian H, that time-evolves vectors in the Hilbert space:

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle \tag{1.25}$$

One key difference between QED and ordinary quantum mechanics is the postulate of fundamental commutators. The term "first quantization" refers to assigning the commutator for position and momentum operators:

$$[x_i, p_j] \equiv x_i p_j - p_j x_i = i\hbar\delta_{ij} \tag{1.26}$$

One may use this postulate to derive the Heisenberg uncertainty relation, the Fourier-transform duality between position and momentum space, and the Schrodinger wave equation [17]. Much of our intuition for first-quantized wave mechanics rests on this commutator.

In quantum field theory (of which QED is a subset), position is demoted to the status of "independent variable," on equal footing with time. We thus discard the commutator (1.26). The classical-field functions ψ and A^μ are then promoted to be *field operators*. (Henceforth in this thesis, we reserve the uppercase $\Psi(x)$ to refer to the electronic field operator, whereas the lowercase $\psi(x)$ continues to refer to a c-number function. Unfortunately, there is no such convention in the literature. The symbol $A^\mu(x)$ will henceforth refer to the photon field operator except when specified otherwise.) In place of (1.26), we postulate the equal-time fermion anticommutators [12, 19–21]

$$\begin{aligned}\left\{\Psi_\alpha(\vec{x},t), \pi_\beta(\vec{x}',t)\right\} &\equiv \Psi_\alpha(\vec{x},t)\pi_\beta(\vec{x}',t) + \pi_\beta(\vec{x}',t)\Psi_\alpha(\vec{x},t) \\ &= i\hbar\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{x}')\end{aligned} \tag{1.27}$$

and

$$\left\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{x}',t)\right\} = \left\{\pi_\alpha(\vec{x},t), \pi_\beta(\vec{x}',t)\right\} = 0 \tag{1.28}$$

where π_β is the canonical conjugate to field Ψ_β, given by

$$\pi_\beta = \frac{\partial \mathscr{L}_{\text{Dirac}}}{\partial \dot{\Psi}_\beta} \tag{1.29}$$

Note that α and β are spinor indices, not to be confused with 4-vector indices. One can show that the Heisenberg equation of motion for Ψ is equivalent to the Dirac equation. A similar procedure is used to construct the quantized radiation and Klein-Gordon field operators, the chief difference being that (1.27) and (1.28) are replaced by commutators. This process of promoting classical fields to quantum field operators is called "second quantization." Appendix B reviews several important details of the quantum field operators and the Hilbert spaces that they act on.

As described previously, the objects Ψ and A^{μ} take on entirely new meanings after second quantization. In particular, $\Psi(x)$ is no longer the probability amplitude (or wave function) $\langle x|\psi\rangle$; instead, $\Psi(x)$ is an abstract operator whose *fundamental* role is to be a building block for the full QED Hamiltonian. This Hamiltonian, which generates the time evolution of QED states according to (1.24), is constructed from the gauge-coupled Lagrangian density (1.21) by the usual Legendre transformation

$$H_{\text{QED}} = \int d^3x \mathscr{H}_{\text{QED}}(x) = \int d^3x \left(\sum_j \pi_j(x) \dot{\Omega}_j(x) - \mathscr{L}_{\text{QED}}(x) \right) \tag{1.30}$$

where $\Omega_j(x)$ is the j^{th} quantum field operator, $\pi_j(x)$ is the j^{th} conjugate momentum operator, and $\mathscr{H}_{\text{QED}}(x)$ is the QED Hamiltonian density [22]. Being constructed from quantized field operators, H_{QED} allows for the creation and annihilation of electrons, positrons, and photons [23].

We thus see that QED does not naively couple single-particle probability currents to classical vector potentials. This discussion shows that the Schrodinger interpretation of quantum mechanics is not equivalent to quantum electrodynamics, although this does not yet answer the radiation question posed in Sec. 1.1.

In Chapter 2, we use coherent states, perturbation theory, and Feynman diagrams to calculate the expected number of photons emitted by a laser-driven electron wave packet, finding that wave-packet size does not matter at any order of perturbation theory. The chapter concludes with a prescription for salvaging the semiclassical scattering amplitude of Sec. 1.2. Chapter 3 treats the incident field non-perturbatively in the Furry picture of QED, demonstrating again that wave-packet size does not influence photoemission, this time in the high-intensity limit. Chapter 4 discusses the importance of assuming the incident light field to be unidirectional, and it contrasts scattering amplitudes for second-quantized and first-quantized matter fields.

Chapter 2

Coherent State Scattering

2.1 Review of Coherent States

To allow for the conceptual possibility of phase-mismatching from a large electron wave packet, we must choose the initial photon state to represent a light pulse with (reasonably) well-defined phase properties. The construction of such a state turns out to be nontrivial, but it provides ample opportunity to gain intuition for the quantized light field. In the remainder of this thesis (excluding Sec. 2.4), we use scaled units such that $\hbar$ and c vanish from the expressions. Our units of electromagnetism remain unrationalized. This breaks the convention of some texts by retaining factors of 4π in the photon field operator and Lagrangian density, thus connecting more easily to the Gaussian unit system used in Chapter 1.

We consider, for now, the free electromagnetic field. The free photon field operator can be written in the Coulomb gauge as

$$\vec{A}(x) = \sum_{\vec{k}} \sqrt{\frac{2\pi}{Vk}} \left[a_{\vec{k}} \hat{\varepsilon}_{\vec{k}} \mathrm{e}^{-ik\cdot x} + a^{\dagger}_{\vec{k}} \hat{\varepsilon}^{*}_{\vec{k}} \mathrm{e}^{ik\cdot x} \right] \tag{2.1}$$

where $\hat{\varepsilon}_{\vec{k}}$ is a polarization vector orthogonal to $\vec{k}$. For simplicity, we suppress the sum over polarizations. The electric field operator is therefore

$$\vec{E}(x) = -\frac{\partial}{\partial t}\vec{A}(x) = i\sum_{\vec{k}}\sqrt{\frac{2\pi k}{V}}\left[a_{\vec{k}}\hat{\varepsilon}_{\vec{k}}\mathrm{e}^{-ik\cdot x} - a_{\vec{k}}^{\dagger}\hat{\varepsilon}_{\vec{k}}^{*}\mathrm{e}^{ik\cdot x}\right] \tag{2.2}$$

The magnetic field operator may similarly be defined. These operators are indeed Hermitian, as observables should be.

The number states presented in Appendix B form a convenient orthonormal basis for the photon Fock space, as they are eigenstates of the free-field Hamiltonian. Unfortunately, the expectation value of the electric field operator

$$\langle\psi|\vec{E}(x)|\psi\rangle \tag{2.3}$$

is zero for all x if $|\psi\rangle$ is a number state. This is true for arbitrarily large photon occupation numbers, holding in spite of the fact that $\left\langle E^2(x)\right\rangle$ is nonzero. As it turns out, number states are highly "nonclassical" in the sense that the correspondence principle does not apply to them in an obvious way; they do, however, serve as a convenient basis for constructing the set of "classical" states, the "coherent states" [24]. This review follows the general approach given in [25] and [26].

2.1.1 Construction of Single-Mode Coherent States

The expression for the operator $\vec{E}(x)$ in (2.2) appears very much like the Fourier expansion of a classical field, except that the Fourier amplitudes are the operators $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$. If these operators were replaced by the scalars $\alpha_{\vec{k}}$ and $\alpha_{\vec{k}}^{*}$ in (2.3), then the quantity $\left\langle\vec{E}(x)\right\rangle$ would indeed be the Fourier expansion of a classical field, with Fourier amplitudes given by those same scalars. This is the motivation for constructing coherent states, which satisfy the eigenvalue equation

$$a_{\vec{k}'}\left|\{\alpha_{\vec{k}}\}\right\rangle = \alpha_{\vec{k}'}\left|\{\alpha_{\vec{k}}\}\right\rangle \tag{2.4}$$

Clearly, if the state $|\psi\rangle$ in (2.3) satisfies the above eigenvalue equation, then $\left\langle\vec{E}(x)\right\rangle$ is a classical field with Fourier expansion coefficients given by $\{\alpha_{\vec{k}}\}$. We need not suppose that $\alpha_{\vec{k}'}$ is necessarily real, as the operator $a_{\vec{k}'}$ is not Hermitian. In fact, $\alpha_{\vec{k}'}$ can be any complex number.

To construct these states, it is helpful to begin in the Hilbert space of a single mode $\vec{k}$. The number states of that mode form a complete basis for this space, so the coherent state $|\alpha\rangle$ can be expressed in terms of them. Hence, (2.4) can be rewritten as

$$a\left(\sum_n |n\rangle\langle n|\alpha\rangle\right) = \alpha\left(\sum_n |n\rangle\langle n|\alpha\rangle\right) \tag{2.5}$$

Linearity and the fact that $a|n\rangle = \sqrt{n}|n-1\rangle$ (with $a|0\rangle = 0$) imply that

$$\sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty} \alpha|n\rangle\langle n|\alpha\rangle \tag{2.6}$$

Re-indexing the left-hand side via $n \to n-1$ and rearranging terms produces

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1}\langle n+1|\alpha\rangle - \alpha\langle n|\alpha\rangle\right)|n\rangle = 0 \tag{2.7}$$

Since the $\{|n\rangle\}$ are linearly independent, the coefficients of each $|n\rangle$ must separately equal zero. The result is that

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}}\langle n|\alpha\rangle \tag{2.8}$$

Computing the first few terms of the series, we find

$$\begin{aligned}
\langle 1|\alpha\rangle &= \frac{\alpha}{\sqrt{1}}\langle 0|\alpha\rangle \\
\langle 2|\alpha\rangle &= \frac{\alpha^2}{\sqrt{2\cdot 1}}\langle 0|\alpha\rangle \\
\langle 3|\alpha\rangle &= \frac{\alpha^3}{\sqrt{3\cdot 2\cdot 1}}\langle 0|\alpha\rangle \\
&\dots \\
\langle n|\alpha\rangle &= \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle
\end{aligned} \tag{2.9}$$

Hence, we find that the coherent state $|\alpha\rangle$ is

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.10}$$

We can compute the normalization constant $\langle 0|\alpha\rangle$ as follows:

$$1 = \langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 \mathrm{e}^{|\alpha|^2} \tag{2.11}$$

Solving for $\langle 0|\alpha\rangle$, we conclude that $\langle 0|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}$, and

$$\begin{aligned} |\alpha\rangle &= \mathrm{e}^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= \mathrm{e}^{-|\alpha|^2/2} \sum_n \frac{(\alpha a^\dagger)^n}{n!} |0\rangle \\ &= \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha a^\dagger} |0\rangle \end{aligned} \tag{2.12}$$

is a normalized, single-mode coherent state.

2.1.2 Properties of Coherent States

We now derive several important properties of coherent states. Up to this point, these states are just mathematical objects defined (in the single-mode case) by (2.12) or (2.4). We have yet to interpret these states physically.

First of all, we can construct multi-mode coherent states by choosing an $\alpha_{\vec{k}}$ for every mode (including $\alpha_{\vec{k}} = 0$ for unoccupied modes, as the vacuum is technically a coherent state with eigenvalue 0); we then use the generalization of (2.12):

$$\left|\{\alpha_{\vec{k}}\}\right\rangle = \prod_{\vec{k}} \left(\mathrm{e}^{-\left|\alpha_{\vec{k}}\right|^2/2} \mathrm{e}^{\alpha_{\vec{k}} a^\dagger_{\vec{k}}} \right) |0\rangle \tag{2.13}$$

By construction, the state $\left|\{\alpha_{\vec{k}}\}\right\rangle$ satisfies the coherent-state eigenvalue equation (2.4) for every $a_{\vec{k}}$. We note that coherent states with finite $\{\alpha_{\vec{k}}\}$ are not generally orthogonal to each other.

The expectation value of the electric field operator is straightforward to compute. We separate the electric field operator into its creation and annihilation parts

$$\vec{E}(x) = \vec{E}^{(+)}(x) + \vec{E}^{(-)}(x) \tag{2.14}$$

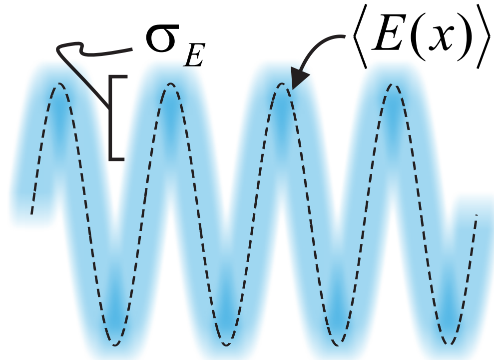


Figure 2.1 The electric field of a single-mode coherent state, including quantum uncertainty σ_E. The dotted line depicts the expectation value of the electric field operator.

where

$$\begin{aligned}\vec{E}^{(+)}(x) &\equiv i\sum_{\vec{k}}\sqrt{\frac{2\pi k}{V}}\hat{\varepsilon}_{\vec{k}}a_{\vec{k}}\mathrm{e}^{-ik\cdot x}\\ \vec{E}^{(-)}(x) &\equiv \left(\vec{E}^{(+)}(x)\right)^{\dagger}\end{aligned} \tag{2.15}$$

Using (2.4) and its adjoint equation, we see that

$$\begin{aligned}\left\langle\vec{E}(x)\right\rangle &= \langle\{\alpha_{\vec{k}}\}|\vec{E}(x)|\{\alpha_{\vec{k}}\}\rangle\\ &= i\sum_{\vec{k}}\sqrt{\frac{2\pi k}{V}}\left[\alpha_{\vec{k}}\hat{\varepsilon}_{\vec{k}}\mathrm{e}^{-ik\cdot x} - \alpha^*_{\vec{k}}\hat{\varepsilon}^*_{\vec{k}}\mathrm{e}^{ik\cdot x}\right]\end{aligned} \tag{2.16}$$

This demonstrates that coherent states of light produce an electric field that, on average, resembles a free classical field. The $\left\{\alpha_{\vec{k}}\right\}$ are Fourier coefficients that determine the structure of $\left\langle\vec{E}(x)\right\rangle$.

The coherent-state uncertainty of the electric field is equal to the uncertainty of the vacuum. This can be shown by using the commutator of $E_i^{(+)}(x)$ and $E_j^{(-)}(x)$, which is

$$\left[E_i^{(+)}(x), E_j^{(-)}(x)\right] = \sum_{\vec{k}}\frac{2\pi k}{V}\varepsilon_{\vec{k}i}\varepsilon^*_{\vec{k}j} \tag{2.17}$$

We find the expectation value of $\vec{E}^2(x)$ in a coherent state to be

$$\begin{aligned}\left\langle \vec{E}\cdot\vec{E}\right\rangle &= \left\langle \vec{E}^{(-)}\cdot\vec{E}^{(-)}+\vec{E}^{(-)}\cdot\vec{E}^{(+)}+\vec{E}^{(+)}\cdot\vec{E}^{(-)}+\vec{E}^{(+)}\cdot\vec{E}^{(+)}\right\rangle \\ &= \left\langle \vec{E}^{(-)}\cdot\vec{E}^{(-)}+\vec{E}^{(-)}\cdot\vec{E}^{(+)}+\vec{E}^{(-)}\cdot\vec{E}^{(+)}+\vec{E}^{(+)}\cdot\vec{E}^{(+)}+\sum_{\vec{k}}\frac{2\pi k}{V}\right\rangle \\ &= \left\langle \vec{E}\right\rangle\cdot\left\langle \vec{E}\right\rangle+\sum_{\vec{k}}\frac{2\pi k}{V}\end{aligned} \tag{2.18}$$

where we have made use of the commutator (2.17) and generalizations of

$$\langle\alpha|a^\dagger a|\alpha\rangle = \alpha^*\alpha = \langle\alpha|a^\dagger|\alpha\rangle\langle\alpha|a|\alpha\rangle \tag{2.19}$$

We see that the variance of $\vec{E}$ in a coherent state is

$$\sigma_E^2 = \left\langle \vec{E}\cdot\vec{E}\right\rangle - \left\langle \vec{E}\right\rangle\cdot\left\langle \vec{E}\right\rangle = \sum_{\vec{k}}\frac{2\pi k}{V} \tag{2.20}$$

which precisely matches that of the vacuum state. The single-mode uncertainty (ie, the square root of the summand) constitutes the “quantum flesh on the classical bones” [26]. Fig. 2.1 illustrates this for a single mode. For large oscillation amplitudes, this quantum flesh becomes negligible, as the correspondence principle would require.

The coherence eigenvalue $\alpha_{\vec{k}'}$ is physically significant in that its magnitude squared determines the expected number of photons in its respective mode. This can be seen via

$$\left\langle n_{\vec{k}'}\right\rangle = \left\langle \{\alpha_{\vec{k}}\}\right| a^\dagger_{\vec{k}'}a_{\vec{k}'}\left|\{\alpha_{\vec{k}}\}\right\rangle = \alpha^*_{\vec{k}'}\alpha_{\vec{k}'} = \left|\alpha_{\vec{k}'}\right|^2 \tag{2.21}$$

The phases of $\{\alpha_{\vec{k}}\}$ determine the relative phases of modes in the Fourier expansion (2.16).

We conclude with a remark that classical charge currents generate coherent states of light. This can be shown by solving for the time-evolution operator generated by the interaction Hamiltonian

$$V(t) = \int d^3x \vec{J}(\vec{x},t)\cdot\vec{A}(\vec{x},t) \tag{2.22}$$

where $\vec{J}$ is an exogenous classical current density. This proof is outlined in [26, 27].

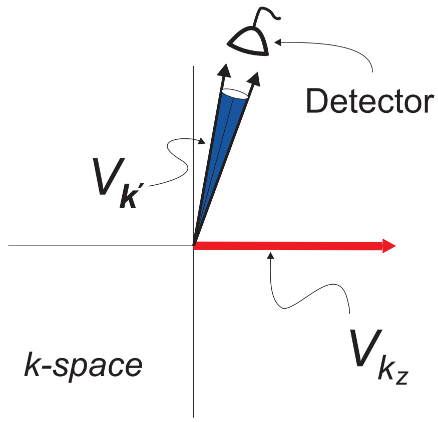


Figure 2.2 Depiction of k-space regions for the incident pulse (V_{k_z}) and photon detector ($V_{\vec{k}'}$).

2.2 Counting Scattered Photons

The matrix elements of the QED scattering operator describe transition probabilities. In and of themselves, these probabilities do not predict *how many* photons will be detected after the interaction is over. To compute that quantity, we must introduce a weighted sum of probabilities.

We begin by identifying two regions of k-space that are of interest, depicted schematically in Fig. 2.2. We define the region V_{k_z} to contain photon momentum vectors comprising the incident light field, which propagates only along the $\hat{z}$-direction. We define the region $V_{\vec{k}'}$ to contain photon momentum vectors that may be intercepted by a detector aligned off-axis (blind to the incident light). The latter region need not be limited to a single ray emanating from the origin, as real photon detectors may subtend a non-vanishing solid angle. The regions V_{k_z} and $V_{\vec{k}'}$ should not be confused with the (position-space) quantization volume V.

Without loss of generality, we suppress spin and polarization indices. In calculating the amount of detected radiation, we are interested in the object

$$\left\langle N_{V_{\vec{k}'}} \right\rangle = \langle \psi(t) | \sum_{V_{\vec{k}'}} a^{\dagger}_{\vec{k}'} a_{\vec{k}'} | \psi(t) \rangle \tag{2.23}$$

This quantity represents the expected number of photons scattered into the region $V_{\vec{k}'}$. The use of

QED scattering theory will require the eventual limit that $t \to \infty$.

We write (2.23) in terms of traditional scattering amplitudes. In the space of states that includes a single electron and an arbitrary number of photons, we can resolve the identity as follows:

$$\begin{aligned}
\mathbb{1} &= \sum_{\vec{p}'} |\vec{p}'\rangle\langle\vec{p}'| \otimes \sum_{\{n_{\vec{k}}\}} |\{n_{\vec{k}}\}\rangle\langle\{n_{\vec{k}}\}| \\
&= \sum_{\vec{p}'} |\vec{p}'\rangle\langle\vec{p}'| \otimes \sum_{\{n_{k_z}\}} \Bigg(|0_{\vec{k}''};\{n_{k_z}\}\rangle\langle 0_{\vec{k}''};\{n_{k_z}\}| + \sum_{\vec{k}''} \left|\vec{k}'';\{n_{k_z}\}\right\rangle\left\langle\vec{k}'';\{n_{k_z}\}\right| \\
&\quad + \sum_{\vec{k}''}\sum_{\vec{k}'''} \left|\vec{k}'',\vec{k}''';\{n_{k_z}\}\right\rangle\left\langle\vec{k}'',\vec{k}''';\{n_{k_z}\}\right| + \dots \Bigg)
\end{aligned} \tag{2.24}$$

where $\{n_{k_z}\}$ represents a configuration of photons in modes $k_z \in V_{k_z}$, and it is understood that $\{\vec{k}'',\vec{k}''',\dots\} \notin V_{k_z}$. This mixture of notations for modes in and out of V_{k_z} will prove useful in the scattering analysis, as it explicitly distinguishes newly-scattered photons from those that were already present in the incident pulse [28]. If we insert this identity between the creation and annihilation operators $a^\dagger_{\vec{k}'}$ and $a_{\vec{k}'}$, we find that

$$a^\dagger_{\vec{k}'} |\vec{p}';0_{\vec{k}''};\{n_{k_z}\}\rangle\langle\vec{p}';0_{\vec{k}''};\{n_{k_z}\}| a_{\vec{k}'} = \left|\vec{p}';\vec{k}';\{n_{k_z}\}\right\rangle\left\langle\vec{p}';\vec{k}';\{n_{k_z}\}\right| \tag{2.25}$$

and

$$a^\dagger_{\vec{k}'} \left|\vec{p}';\vec{k}'';\{n_{k_z}\}\right\rangle\left\langle\vec{p}';\vec{k}'';\{n_{k_z}\}\right| a_{\vec{k}'} = \left|\vec{p}';\vec{k}',\vec{k}'';\{n_{k_z}\}\right\rangle\left\langle\vec{p}';\vec{k}',\vec{k}'';\{n_{k_z}\}\right| \tag{2.26}$$

for $\vec{k}'' \neq \vec{k}'$, and

$$a^\dagger_{\vec{k}'} \left|\vec{p}';\vec{k}'';\{n_{k_z}\}\right\rangle\left\langle\vec{p}';\vec{k}'';\{n_{k_z}\}\right| a_{\vec{k}'} = \sqrt{2}\,|\vec{p}';2_{\vec{k}'};\{n_{k_z}\}\rangle\langle\vec{p}';2_{\vec{k}'};\{n_{k_z}\}|\,\sqrt{2} \tag{2.27}$$

for $\vec{k}'' = \vec{k}'$. (The pattern for higher-order terms should be clear.) The detected photon number may therefore be written as

$$\begin{aligned}
\left\langle N_{V_{\vec{k}'}}\right\rangle = \sum_{\vec{p}'}\sum_{V_{\vec{k}'}}\sum_{\{n_{k_z}\}} \Bigg[\left|\left\langle\vec{p}';\vec{k}';\{n_{k_z}\}|\psi(t)\right\rangle\right|^2 + \sum_{\vec{k}''\neq\vec{k}'} \left|\left\langle\vec{p}';\vec{k}',\vec{k}'';\{n_{k_z}\}|\psi(t)\right\rangle\right|^2 \\
+2\left|\langle\vec{p}';2_{\vec{k}'};\{n_{k_z}\}|\psi(t)\rangle\right|^2 + \dots \Bigg]
\end{aligned} \tag{2.28}$$

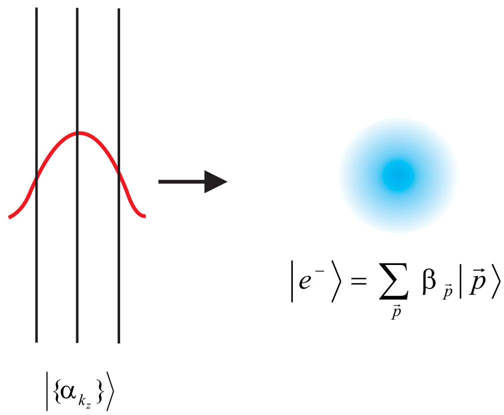


Figure 2.3 Depiction of the initial condition of a unidirectional pulse and an electron wave packet. The straight lines represent (infinite) planar wavefronts.

We see explicitly that the state $|\psi(t)\rangle$ is projected onto a single basis vector *before* squaring and summing over the states of that basis. This is in agreement with the probability interpretation of quantum mechanics [29], where (2.28) is a weighted sum of the probabilities of scattering photons into the k-space region $V_{\vec{k}'}$.

2.3 Scattering of Coherent Light States

We are now prepared to compute the average number of photons scattered to a detector that is aligned off-axis to the incident photon beam. Let the initial state of the system (before interaction) be represented by the disentangled state

$$|\psi_{in}\rangle = \left(\sum_{\vec{p}} \beta_{\vec{p}} \, |\vec{p}\rangle\right) \otimes \left|\{\alpha_{k_z}\}\right\rangle = \sum_{\vec{p}} \beta_{\vec{p}} \left|\vec{p}; \{\alpha_{k_z}\}\right\rangle \tag{2.29}$$

where $\{\alpha_{k_z}\}$ are chosen to represent a unidirectional light pulse. Note that only modes $k_z \in V_{k_z}$ are initially occupied in the light field. The coefficients $\{\beta_{\vec{p}}\}$ can be chosen to construct an arbitrary (potentially large) free electron wave packet. Fig. 2.3 depicts this initial condition.

We time-evolve this state in the interaction picture [23] using the scattering operator S. This operator is the interaction-picture version of $U(\infty, -\infty)$, the time-evolution operator (1.25). The

Dyson expansion for S is

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n T[\mathscr{H}_{\text{int}}(x_1)\ldots\mathscr{H}_{\text{int}}(x_n)] \tag{2.30}$$

where

$$\mathscr{H}_{\text{int}}(x) = e : \bar{\Psi}(x)\gamma_\mu \Psi(x) A^\mu(x) : \tag{2.31}$$

is the normally-ordered QED interaction Hamiltonian density, T is the time-ordering operator, and $A^\mu(x)$ and $\Psi(x)$ are the standard free-field operators for photons and Dirac electrons/positrons, respectively [12]. We include expressions for these operators in Appendix B.

As shown in equation (2.28), we must compute and then square amplitudes of the form

$$\langle \vec{p}'; \vec{k}', \vec{k}'', \ldots; \{n_{k_z}\} | S | \psi_{in} \rangle \tag{2.32}$$

where primed wave vectors represent photons scattered outside of V_{k_z}. We emphasize that the parameters defining the bra vector are fixed before squaring. To properly characterize the Feynman diagrams that contribute to these amplitudes, we must examine the general framework (not the fine details) of the relevant Wick expansion of (2.30). Wick's Theorem rewrites the time-ordered operator products in (2.30) as sums of normally-ordered operator products [21]. We find (after some algebra) that

$$\begin{aligned} \langle \vec{p}'; \vec{k}', \vec{k}'', \ldots; \{n_{k_z}\} | S | \psi_{in} \rangle = & \sum_{n=2}^{\infty} \sum_{\vec{p}} \beta_{\vec{p}} \frac{(-ie)^n}{n!} \int d^4x_1 \ldots d^4x_n \times \\ & \sum_{\xi} C_\xi S_F(x_{\xi_1}, x_{\xi_2}) \ldots S_F(x_{\xi_{n-1}}, x_{\xi_n}) \times \\ & \sum_{0 \le l \le n-2} \sum_{\zeta} D(x_{\zeta_1}, x_{\zeta_2}) \ldots D(x_{\zeta_{l-1}}, x_{\zeta_l}) \times \\ & \langle \vec{p}'; \vec{k}', \vec{k}'', \ldots; \{n_{k_z}\} | \bar{\Psi}^{(-)}(x_{\xi_n}) : A(x_{\zeta_{l+1}}) \ldots A(x_{\zeta_n}) : \Psi^{(+)}(x_{\xi_1}) | \vec{p}; \{\alpha_{k_z}\} \rangle \end{aligned} \tag{2.33}$$

where ξ represents a particular set of $n-1$ contractions of fermion operators, ζ represents a set of contractions of an *even* number l of photon operators, and C_ξ contains all gamma matrices and any constants that arise from fermion contraction ξ. All polarization, spin, and spinor/gamma matrix indices have been suppressed. The functions $S_F(x,x')$ and $D(x,x')$ represent fermion and photon propagators, respectively. The photon propagators introduce radiative corrections, which, among other terms, require renormalization for explicit calculation. This does not affect our analysis. We note that (2.33) is valid only as an asymptotic series in n [30, 31].

We will not compute any terms of (2.33) explicitly, although a few comments are in order. Since $\vec{k}'$, $\vec{k}''$, etc do not belong to V_{k_z}, there must be a creation operator $A^{(-)}(x_i)$ (defined consistently with (2.15)) for every primed photon to 'create' that state from the initial one (or else the amplitude would vanish from orthogonality between the bra and the ket). A similar argument can show that all matter operators $\bar{\Psi}$ and Ψ must be contracted except for the two that annihilate and create the incoming and outgoing electron states; hence, there are $n-1$ fermion contractions. It can be shown kinematically that $\{\vec{k}',\vec{k}'',\ldots\} \notin V_{k_z}$ implies that $\vec{p}' \neq \vec{p}$ in non-vanishing diagrams. These arguments indicate that certain types of intuitively-plausible Feynman diagrams vanish trivially. Fig. 2.4 shows a generic non-vanishing Feynman diagram. The external lines referring to primed quantities are fixed before squaring, as demonstrated by (2.28).

For every field operator that is not contracted, there is an external particle line [23]. All $A^{(+)}(x)$ operators appear to the right, owing to normal-ordering. Acting on the coherent state, they repeatedly pull out the (c-number) eigenvalue

$$A^{(+)}_{\{\alpha_{k_z}\}}(x) \equiv \sum_{k_z \in V_{k_z}} \sqrt{\frac{2\pi}{k_z V}}\, \alpha_{k_z} \varepsilon_{k_z} e^{-ik_z \cdot x} \tag{2.34}$$

without changing the state. We note that each operator $A^{(+)}(x_i)$ produces a different sum $A^{(+)}_{\{\alpha_{k_z}\}}(x_i)$ with its own summation index $k_z^{(i)}$. This feature will be important to our analysis. All $A^{(-)}$ operators appear on the left. Some of them produce the scattered photons $\vec{k}'$, $\vec{k}''$, etc, while the remainder produce photons that are forward-scattered into V_{k_z}. In the usual manner, they contribute complex

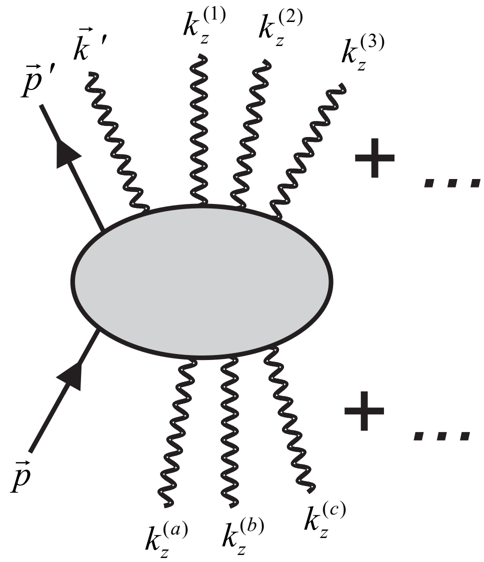


Figure 2.4 Generic Feynman diagram showing possible external lines. Time is assumed to run upward.

exponentials of the form

$$e^{ik'\cdot x}, e^{ik''\cdot x}, \ldots \tag{2.35}$$

for photons scattered outside of V_{k_z}, and

$$\sum_{k_z \in V_{k_z}} g\left(\{n_{k_z}\}\right) \sqrt{\frac{2\pi}{k_z V}} \varepsilon^*_{k_z} e^{ik_z \cdot x} \tag{2.36}$$

for photons forward-scattered into V_{k_z}. The items (2.34), (2.35), and (2.36) designate the external photon lines of Feynman diagrams. In typical low-order calculations, the external lines are determined uniquely by the initial state (ket) and the projection (bra). That is clearly not the case when considering coherent states, especially for high-order terms in the expansion. The electron may in principle absorb an arbitrary number of photons from V_{k_z} (dictated by the number of $A^{(+)}(x)$ operators in the product) or forward-scatter as many photons as are allowed by the final projection onto $\left\langle \{n_{k_z}\} \right|$. This feature, along with the true arbitrariness of our momentum distributions, causes the present scenario to deviate from previous packet-packet calculations [18, 32]. Our approach does, however, rely on similar kinematic principles.

The integrations over $d^4x_1...d^4x_n$ produce delta functions that enforce energy-momentum conservation at every vertex. These delta functions allow for the evaluation of many of the momentum-space integrals that compose the electron and photon propagators in (2.33). When the smoke clears, there remains (for each summed term of (2.33)) a single four-dimensional delta function that enforces energy-momentum conservation of the external lines. (Three of the delta functions are of the Kronecker variety if we quantize in volume V, although this does not change the arguments that follow.) These kinematic constraints are well known and constitute one of the Feynman rules for evaluation of transition amplitudes [12, 18, 33]. Ignoring numerical factors, the complex exponentials in the previous paragraph indicate that (2.33) must include delta functions of the form

$$\delta^{(4)}\left(p'+k'+k''+\ldots+k_z^{(1)}+k_z^{(2)}+\ldots-k_z^{(a)}-k_z^{(b)}-\ldots-p\right), \tag{2.37}$$

where, as in Fig. 2.4, numerical superscripts indicate forward-emitted photons and letter superscripts indicate photons absorbed from the incident light. It appears, at first glance, that the square of the amplitude (2.33) might include cross terms between different electron momenta as well as different photon momenta, as a single four-delta cannot collapse the many sums in (2.33).

A careful examination of the kinematic constraints enforced by (2.37) demonstrates that the scattering does not depend on the relative phases of the momenta that compose the initial electron wave packet. We remind the reader that, in the amplitude (2.33), the momenta of all primed external lines (belonging to the bra) are fixed before the amplitude is squared. If the incident light pulse is unidirectional, then the kinematic constraints make the scattering amplitude (2.33) zero except when

$$\begin{aligned}
&p'_{(x)}+k'_{(x)}+k''_{(x)}+\ldots=p_{(x)}\\
&p'_{(y)}+k'_{(y)}+k''_{(y)}+\ldots=p_{(y)}\\
&p'_{(z)}+k'_{(z)}+k''_{(z)}+\ldots+k_z^{(1)}+k_z^{(2)}+\ldots=k_z^{(a)}+k_z^{(b)}+\ldots+p_{(z)}\\
&E_{\vec{p}'}+k'+k''+\ldots+\left|k_z^{(1)}\right|+\left|k_z^{(2)}\right|+\ldots=\left|k_z^{(a)}\right|+\left|k_z^{(b)}\right|+\ldots+E_{\vec{p}}
\end{aligned} \tag{2.38}$$

The x and y constraints collapse two dimensions out of the sum over $\vec{p}$. Since the incident pulse is unidirectional, we have $k_z = |k_z|$ for all $k_z \in V_{k_z}$. Then both of the bottom two constraints contain the identical quantity $k_z^{(a)} + k_z^{(b)} + \ldots - k_z^{(1)} - k_z^{(2)} - \ldots$, which can be substituted between them. This results in

$$p'_{(z)} + k'_{(z)} + k''_{(z)} + \ldots = E_{\vec{p}'} + k' + k'' + \ldots - E_{\vec{p}} + p_{(z)} \tag{2.39}$$

This constraint must be the same for *every* nonzero contribution to (2.33) (to all orders of perturbation theory), as the substitution of momenta from V_{k_z} can always be made for a unidirectional pulse. This final constraint, along with the simpler ones in the x and y directions, entirely determines the value of $\vec{p} = \tilde{\vec{p}}$ for which the amplitude (2.33) is nonzero. Thus, kinematic constraints collapse the sum over $\vec{p}$, and the amplitude-squared of (2.33) depends on $\beta_{\vec{p}}$ only via

$$\left|\left\langle \vec{p}'; \vec{k}', \vec{k}'', \ldots; \{n_{k_z}\} | S | \psi_{in} \right\rangle\right|^2 \propto \left|\beta_{\tilde{\vec{p}}}\right|^2 \tag{2.40}$$

That is, the relative phases of $\{\beta_{\vec{p}}\}$ have no influence on the scattered radiation.

The relative phases of $\{\beta_{\vec{p}}\}$ play a key role in determining the spatial size of an electron wave packet. A simple change of these phases such as

$$\beta_{\vec{p}} \to \beta_{\vec{p}} e^{-iE_{\vec{p}}T} \tag{2.41}$$

accounts for the natural quantum spreading that characterizes free-particle dynamics. This spreading can drastically change the spatial scale of a wave packet from being almost point-like (relative to the wavelength of the stimulating field) to spanning many wavelengths. We have shown that such transformations have no effect on the scattered radiation; that is, size doesn't matter.

Once the sum over $\vec{p}$ is collapsed, there remains only a single delta function. This delta function determines the precise value that $k_z^{(a)} + k_z^{(b)} + \ldots - k_z^{(1)} - k_z^{(2)} - \ldots$ must take for the amplitude to be nonzero. This suggests that absorption and re-emission of multiple photons into V_{k_z} can effectively be treated kinematically as the absorption/emission of an single unidirectional photon of momentum

$$\Delta k_z = k_z^{(a)} + k_z^{(b)} + \ldots - k_z^{(1)} - k_z^{(2)} - \ldots \tag{2.42}$$

The final delta function does not collapse all of the remaining sums over V_{k_z}. This indicates that the relative phases of $\{\alpha_{k_z}\}$ do matter. This result is unsurprising, however. The relative phases of $\{\alpha_{k_z}\}$ can determine the incident light's state of chirp, for example. Rearrangement of those phases can change the temporal profile of the pulse from short to long without changing the spectral content. This can drastically affect the instantaneous intensity observed by the electron, thereby altering nonlinear radiative transitions.

2.4 Adapting First-Quantized Amplitudes

In Chapter 1, we showed that first-quantized quantum mechanics does not dictate the form of the classical scattered field A_s^μ, except possibly by the gauge coupling discussed in Sec. 1.3. We now show how to intelligently choose A_s^μ in order to match the first-order amplitude (1.13) with the corresponding result from quantum electrodynamics. For the sake of comparison with the semiclassical result, this section is written in Gaussian units and includes the polarization index λ.

The lowest-order term of (2.28) that contributes to $\left\langle N_{V_{\vec{k}'}} \right\rangle$ is

$$\left| \left\langle \vec{p}'; \vec{k}'\lambda'; \{n_{k_z}\} | S^{(1)} | \psi_{\text{in}} \right\rangle \right|^2 \tag{2.43}$$

Since we are now working with a spinless (scalar) electron, the interaction Hamiltonian density changes to accommodate scalar fields. This interaction, derived from the gauge-coupled Lagrangian density (1.22), is effectively

$$\mathscr{H}_{\text{int}} =: \frac{iec}{\hbar} \left[\phi^\dagger \left(\partial_\mu \phi \right) - \left(\partial_\mu \phi^\dagger \right) \phi \right] A^\mu - \frac{e^2}{\hbar^2} A_\mu A^\mu \phi^\dagger \phi : \tag{2.44}$$

where ϕ is the scalar field operator

$$\phi(x) = \sum_{\vec{p}} \sqrt{\frac{\hbar^2}{2E_p V}} \left(\mathrm{b}_{\vec{p}} \mathrm{e}^{-ip\cdot x} + \mathrm{d}_{\vec{p}}^\dagger \mathrm{e}^{ip\cdot x} \right) \tag{2.45}$$

and A^μ is the photon field operator

$$A^\mu(x) = \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi\hbar c}{Vk}} \left(a_{\vec{k}\lambda} \varepsilon^\mu_{\vec{k}\lambda} \mathrm{e}^{-ik\cdot x} + a^\dagger_{\vec{k}\lambda} \varepsilon^{\mu *}_{\vec{k}\lambda} \mathrm{e}^{ik\cdot x} \right) \tag{2.46}$$

expressed in Gaussian units. As all particles here are bosons, the a, b, and d operators satisfy usual bosonic commutation relations with their adjoints (see (B.4)). In a technical sense, (2.44) should include several non-covariant terms, but it can be shown that those terms do not contribute to scattering amplitudes [32]. Note the normal-ordering of (2.44).

The scattering operator becomes

$$\begin{aligned} S^{(1)} &= -\frac{i}{\hbar c} \int d^4x \mathscr{H}_{\text{int}}(x) \\ &= -\frac{i}{\hbar c} \int d^4x : \left\{ \frac{iec}{\hbar} \left[\phi^\dagger \left(\partial_\mu \phi\right) - \left(\partial_\mu \phi^\dagger\right) \phi \right] A^\mu - \frac{e^2}{\hbar^2} A_\mu A^\mu \phi^\dagger \phi \right\} : \\ &= -\frac{i}{\hbar c} \int d^4x : \left\{ \frac{iec}{\hbar} 2A^\mu \phi^\dagger \left(\partial_\mu \phi\right) - \frac{e^2}{\hbar^2} A_\mu A^\mu \phi^\dagger \phi \right\} : \end{aligned} \tag{2.47}$$

where we've exploited the hermiticity of $i\partial_\mu$ and the fact that the operator $\partial_\mu A^\mu$ effectively vanishes when A^μ is quantized in the Lorenz gauge. The matrix element

$$\left\langle \vec{p}'; \vec{k}'\lambda'; \{n_{k_z}\} | S^{(1)} | \psi_{\text{in}} \right\rangle = \sum_{\vec{p}} \beta^{(0)}_{\vec{p}} \left\langle \vec{p}'; \vec{k}'\lambda'; \{n_{k_z}\} | S^{(1)} | \vec{p}; \{\alpha_{k_z}\} \right\rangle \tag{2.48}$$

is now straightforward to compute. We find

$$\begin{aligned} \langle \vec{p}' | : \phi^\dagger \phi : | \vec{p} \rangle &= \frac{\hbar^2}{2} \left(\frac{1}{\sqrt{E_{p'} V}} \mathrm{e}^{ip'\cdot x} \right) \left(\frac{1}{\sqrt{E_p V}} \mathrm{e}^{-ip\cdot x} \right) \\ \langle \vec{p}' | : \phi^\dagger \left(\partial_\mu \phi\right) : | \vec{p} \rangle &= \frac{\hbar^2}{2} \left(\frac{1}{\sqrt{E_{p'} V}} \mathrm{e}^{ip'\cdot x} \right) \partial_\mu \left(\frac{1}{\sqrt{E_p V}} \mathrm{e}^{-ip\cdot x} \right) \end{aligned} \tag{2.49}$$

for the electronic portions and

$$\begin{aligned} \langle \vec{k}'\lambda'; \{n_{k_z}\} | A^\mu | \{\alpha_{k_z}\} \rangle &= \langle \{n_{k_z}\} | \{\alpha_{k_z}\} \rangle \left(\sqrt{\frac{2\pi\hbar c}{Vk'}} \varepsilon^{\mu *}_{\vec{k}'\lambda'} \mathrm{e}^{ik'\cdot x} \right) \\ \langle \vec{k}'\lambda'; \{n_{k_z}\} | : A\cdot A : | \{\alpha_{k_z}\} \rangle &= 2 \langle \{n_{k_z}\} | \{\alpha_{k_z}\} \rangle \left(\sqrt{\frac{2\pi\hbar c}{Vk'}} \varepsilon^{*}_{\vec{k}'\lambda'} \mathrm{e}^{ik'\cdot x} \right) \cdot A^{(+)}_{\{\alpha_{k_z}\}}(x) \end{aligned} \tag{2.50}$$

for the photonic portions, where $A^{(+)}_{\{\alpha_{k_z}\}}(x)$ is the eigenvalue (2.34) of the photon annihilation operator $A^{(+)}(x)$ corresponding to the coherent state $\left|\{\alpha_{k_z}\}\right\rangle$. Combining terms and rearranging constants, we find that

$$\left\langle \vec{p}\,';\vec{k}'\lambda';\{n_{k_z}\}|S^{(1)}|\psi_{\text{in}}\right\rangle = \frac{i}{2\hbar mc}\left\langle\{n_{k_z}\}|\{\alpha_{k_z}\}\right\rangle\sum_{\vec{p}}\beta^{(0)}_{\vec{p}}\int d^4x\,\psi^*_{\vec{p}\,'}V_{\text{int}}\psi_{\vec{p}} \tag{2.51}$$

where we define

$$V_{\text{int}} \equiv -\frac{2ie}{c}\left(\sqrt{\frac{2\pi\hbar c}{Vk'}}\varepsilon^*_{\vec{k}'\lambda'}\mathrm{e}^{ik'\cdot x}\right)\cdot\partial + \frac{2e^2}{c^2}A^{(+)}_{\{\alpha_{k_z}\}}(x)\cdot\left(\sqrt{\frac{2\pi\hbar c}{Vk'}}\varepsilon^*_{\vec{k}'\lambda'}\mathrm{e}^{ik'\cdot x}\right) \tag{2.52}$$

and

$$\psi_{\vec{p}} \equiv \sqrt{\frac{mc^2}{E_pV}}\mathrm{e}^{-ip\cdot x} \tag{2.53}$$

Note that (2.51) and (2.52) have been written in suggestive notation, reminiscent of the first-quantized expressions (1.13) and (1.11), respectively.

It is interesting to note that, in spite of their resemblance, (2.51) and (1.13) are fundamentally different quantities. The semiclassical amplitude, when squared, represents a probability for a *single* electron, whereas the square of the QED amplitude represents a *combined* electron-photon probability. The remarkable feature here is that an *ad hoc* prescription for the classical field A^μ_s can make these amplitudes match. In this way, the semiclassical amplitude is interpreted as a multi-particle quantity [15].

By comparing the scalar QED amplitude with the semiclassical amplitude, it becomes clear that we can appropriately choose the scattered field $A^\mu_s(x)$ to be the negative-frequency component of a plane wave with energy $\hbar ck'$:

$$A^\mu_s(x) = \sqrt{\frac{2\pi\hbar c}{Vk'}}\varepsilon^{\mu *}_{\vec{k}'\lambda'}\mathrm{e}^{ik'\cdot x} \tag{2.54}$$

Likewise, it is also appropriate to choose only the positive-frequency component of the classical incident field

$$\begin{aligned} A_i^\mu(x) &= A^{(+)}_{\{\alpha_{k_z}\}}(x) \\ &= \sum_{k_z\lambda_z} \alpha_{k_z} \sqrt{\frac{2\pi\hbar c}{Vk_z}} \varepsilon^\mu_{k_z\lambda_z} \mathrm{e}^{-ik_z\cdot x} \end{aligned} \tag{2.55}$$

As a practical matter, either of these fields may be taken to be real valued, but *not* both.

If both of these fields are taken to be real, then the integration of complex exponentials in (2.51) produces four sets of delta functions, only one of which enforces energy-momentum conservation. Two of the extraneous sets of delta functions

$$\delta^4\left(p'+k'+k_z-p\right) \quad \text{and} \quad \delta^4\left(p'-k'-k_z-p\right) \tag{2.56}$$

vanish because their regions of support are disjoint. The last set of extraneous delta functions switches the incident and scattered photons:

$$\delta^4\left(p'+k_z-p-k'\right) \tag{2.57}$$

This term does not necessarily vanish and, hence, yields errors that cause the semiclassical amplitude to deviate from the QED amplitude. If the scattered field is given precisely by (2.54), then this non-vanishing extraneous term does not appear. The bizarre conclusion reached by this analysis is that this first-quantized radiation theory (for a single-electron source) is manifestly incorrect if the "scattered" field is considered to be real valued. This is merely one indication that it is conceptually advantageous to adopt the language and perspective of quantum electrodynamics.

One small difference between (2.51) and (1.13) is that the semiclassical result treats the incident field non-perturbatively via Volkov functions. Our QED result treats the incident field perturbatively and, thus, uses plane waves. This discrepancy could be remedied by quantizing in the Furry picture of QED, which expands the field operator $\phi(x)$ using Volkov functions as a basis. We do this for Dirac particles in Chapter 3.

Another difference between the semiclassical and QED amplitudes is that (2.51) is proportional to the factor $\langle\{n_{k_z}\}|\{\alpha_{k_z}\}\rangle$. When computing probabilities in the state space $\{\vec{p}\,',\vec{k}'\lambda'\}$, we should sum over the unobserved, forward-scattered photons. If this is done, the factor disappears because

$$\sum_{\{n_{k_z}\}} \left|\langle\{n_{k_z}\}|\{\alpha_{k_z}\}\rangle\right|^2 = 1 \tag{2.58}$$

owing to completeness. Therefore, we can ignore this factor under the assumption that the sum over forward-scattered photons has already been done.

If we take this semiclassical picture seriously, it seems that our choice for A_s^μ suggests the repugnant notion that the scattered field is plane wave in nature. This rightly seems at odds with the fact that the outgoing photon is undoubtedly some kind of packet. If the stimulating light has compact temporal support, then depending on distances involved, one would expect a photodetector monitoring scattered photons to click within a certain time interval (in the event that there is a click). On the other hand, a single-mode plane wave is unable to specify a time window.

The QED picture indicates that the outgoing photon is indeed a packet. When calculating probabilities for observable measurements, one must project the normalized state onto the eigenbasis of the measurement. These projection amplitudes are *first* squared and *then* summed over a subset of the basis eigenvalues [29]. The final state of the photon-electron system (a packet) is given by $S|\psi_{\text{in}}\rangle$, which we project onto a plane-wave basis before squaring. (One might project this state onto any other observable basis, but momentum eigenvectors make the kinematics of the interaction transparent.)

Introducing the single-mode potential (2.54) as a perturbation in a semiclassical picture is typical [12, 15], and it produces the effect of the *projection* described above. In the literature, it is common to refer to (2.54) as the "emitted photon," but this is somewhat of a misnomer. Prior to the measurement, many momenta may be present in the scattered field. Projecting onto a plane-wave basis, however, allows one to connect momentum measurements with calculable probabilities.

Chapter 3

Furry Picture

3.1 Quantization

The preceding analysis via QED perturbation theory is valid for a broad range of incident intensities, but it breaks down for ultra-intense beams [14]. To characterize the emission of radiation in this regime, we must treat the incident field non-perturbatively.

We begin at the launchpad of second quantization, the Lagrangian density (1.21). We separate the interaction Lagrangian density as follows:

$$\mathscr{L}_{\text{int}}(x) = -e\bar{\Psi}(x)\gamma_\mu\Psi(x)\left[A^\mu(x) + A^\mu_{ext}(x)\right] \tag{3.1}$$

where $A^\mu_{ext}(x)$ represents the classical external potential (a c-number function) and $A^\mu(x)$ is the free photon field operator [34]. In the Furry picture [35], we absorb the interaction with the external field into the "free" electronic Lagrangian density:

$$\mathscr{L}_{\text{Dirac}} = \bar{\Psi}\left(i\gamma\cdot\partial - m\right)\Psi \quad \rightarrow \quad \mathscr{L}_L = \bar{\Psi}_L\left(i\gamma\cdot\partial - e\gamma\cdot A_{ext} - m\right)\Psi_L \tag{3.2}$$

The quantized fields must therefore satisfy

$$\left(i\gamma\cdot\partial - e\gamma\cdot A_{ext} - m\right)\Psi_L = 0 \tag{3.3}$$

We use the subscript L to denote operators that are laser-dressed.

We return now to the assumption that the incident light field is unidirectional. If $A^{\mu}_{ext}(x)$ is a function of only $\eta \equiv n \cdot x = t - \hat{n} \cdot \vec{x}$, then the Volkov functions $\{\psi^{\nu}_{\vec{p}r}\}$ [14] are a solution basis for (3.3). Explicitly, these c-number solutions are

$$\psi^{\nu}_{\vec{p}r}(x) = \sqrt{\frac{m}{V|E_p|}} \left[1 + \frac{e}{2p \cdot n} \gamma \cdot n \gamma \cdot A_{ext}(\eta) \right] u_{\vec{p}r} e^{-ip \cdot x - i \int_{-\infty}^{\eta} S(\eta') d\eta'} \tag{3.4}$$

where

$$S(\eta') = \frac{ep \cdot A_{ext}(\eta')}{p \cdot n} - \frac{e^2 A_{ext}(\eta') \cdot A_{ext}(\eta')}{2p \cdot n} \tag{3.5}$$

and the $u_{\vec{p}r}$ are the free-particle Dirac spinors satisfying

$$(\gamma \cdot p - m) u_{\vec{p}r} = 0. \tag{3.6}$$

The index r specifies one of the four spinor solutions of (3.6). Two of these are negative-energy solutions, such that $p^0 < 0$. In Appendix C, we prove that (3.4) is a solution to the Dirac equation, with the further requirement that the potential satisfy the covariant analog of transversality, $n \cdot A_{ext} = 0$. It can be shown [36] that the Volkov functions are orthogonal, such that

$$\int d^3x \psi^{\nu\dagger}_{\vec{p}'r'}(x) \psi^{\nu}_{\vec{p}r}(x) = \delta_{\vec{p}'\vec{p}} \delta_{r'r} \tag{3.7}$$

It is tempting to associate the parameter $\vec{p}$ with particle 'momentum', since these functions become plane waves in the limit $A_{ext} \to 0$. However, this association is only weak, the reason being that $\vec{p}$ is not a conserved quantity when A_{ext} is nonzero [13]. This fact becomes apparent in scattering calculations. One may describe $\vec{p}$ as an 'asymptotic' momentum in the sense that the function $\psi^{\nu}_{pr}(x)$ asymptotically approaches a plane wave as $t \to \pm\infty$ if the stimulating field is a pulse.

We expand the dressed matter field operator in the basis of Volkov functions:

$$\Psi_L(x) = \sum_{\vec{p}r} \mathrm{b}_{\vec{p}r} \psi^{\nu}_{\vec{p}r}(x) \tag{3.8}$$

where the $\{\mathrm{b}_{\vec{p}r}\}$ are operator-valued coefficients. The momentum field conjugate to $\Psi_L(x)$ is

$$\pi_L(x) = \frac{\partial \mathscr{L}_L}{\partial \dot{\Psi}_L} = i\bar{\Psi}_L\gamma^0 = i\Psi_L^\dagger \tag{3.9}$$

As is customary in canonical quantization, we second-quantize the field by imposing fermionic anticommutation relations between the field Ψ_L and its conjugate momentum π_L:

$$\begin{aligned} \{\Psi_{L\alpha}(\vec{x},t), \pi_{L\beta}(\vec{x}',t)\} &= \left\{\Psi_{L\alpha}(\vec{x},t), i\Psi^\dagger_{L\beta}(\vec{x}',t)\right\} = i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{x}') \\ \{\Psi_{L\alpha}(\vec{x},t), \Psi_{L\beta}(\vec{x}',t)\} &= \{\pi_{L\alpha}(\vec{x},t), \pi_{L\beta}(\vec{x}',t)\} = 0 \end{aligned} \tag{3.10}$$

where α and β denote spinor components in this context. We can solve for $\mathrm{b}_{\vec{p}r}$ in (3.8) by exploiting the orthonormality of the Volkov functions. The result is

$$\mathrm{b}_{\vec{p}r} = \int d^3x \psi^{v\dagger}_{\vec{p}r}(x)\Psi_L(x) \tag{3.11}$$

For the adjoint, we find that

$$\mathrm{b}^\dagger_{\vec{p}'r'} = \int d^3x' \Psi^\dagger_L(x')\psi^v_{\vec{p}'r'}(x') \tag{3.12}$$

We can calculate the equal-time anticommutation relation for $\mathrm{b}_{\vec{p}r}$ and $\mathrm{b}_{\vec{p}'r'}$ from (3.10):

$$\begin{aligned} \left\{\mathrm{b}_{\vec{p}r}, \mathrm{b}^\dagger_{\vec{p}'r'}\right\} &= \left\{\int d^3x \psi^{v\dagger}_{\vec{p}r}(x)\Psi_L(x), \int d^3x' \Psi^\dagger_L(x')\psi^v_{\vec{p}'r'}(x')\right\} \\ &= \int d^3x \psi^{v\dagger}_{\vec{p}r\alpha}(x) \int d^3x' \psi^v_{\vec{p}'r'\beta}(x') \left\{\Psi_{L\alpha}(x), \Psi^\dagger_{L\beta}(x')\right\} \\ &= \int d^3x \psi^{v\dagger}_{\vec{p}r\alpha}(x) \int d^3x' \psi^v_{\vec{p}'r'\beta}(x') \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{x}') \\ &= \delta_{\vec{p}\vec{p}'}\delta_{rr'} \end{aligned} \tag{3.13}$$

where we have taken advantage of the bilinearity of the anticommutator and the orthonormality of the Volkov functions. The operators $\mathrm{b}^\dagger$ and b respectively create and annihilate particles in Volkov states. The anticommutation relation (3.13) assures that these particles satisfy the characteristic antisymmetry of fermions.

Lastly, we interpret the negative-energy states as antiparticles. Separating out the negative-energy part of (3.8), we have

$$\Psi_L(x) = \sum_{\vec{p}} \sum_{r=1,2} \mathrm{b}_{\vec{p}r}\psi^v_{\vec{p}r}(x) + \sum_{\vec{p}} \sum_{r=3,4} \mathrm{b}_{\vec{p}r}\psi^v_{\vec{p}r}(x) \tag{3.14}$$

Following the convention of [12], we redefine the $r = 3,4$ objects

$$
\begin{aligned}
\mathrm{d}^{\dagger}_{\vec{p}1} &\equiv -\mathrm{b}_{-\vec{p}4} \\
\mathrm{d}^{\dagger}_{\vec{p}2} &\equiv \mathrm{b}_{-\vec{p}3} \\
v_{\vec{p}1} &\equiv -u_{-\vec{p}4} \\
v_{\vec{p}2} &\equiv u_{-\vec{p}3}
\end{aligned}
\tag{3.15}
$$

These (*ad hoc*) definitions are motivated by the requirement that the Hamiltonian's eigenvalues be bounded from below, thus preventing infinite downward transitions. They are reminiscent of Dirac's hole theory, in which positrons are the physical manifestations of unoccupied negative-energy electron states [16]. From this perspective, the annihilation of a negative-energy electron of momentum $-\vec{p}$ and spin *down* appears (via energy, momentum, charge, and spin conservation) as the creation of a positive-energy positron with momentum $+\vec{p}$ and spin *up*. The shuffling of minus signs is due to a symmetry of charge conjugation. We redefine the fermionic vacuum to be the absence of b- and d-type particles. The definitions in (3.15) indicate that d and $\mathrm{d}^{\dagger}$ satisfy the same anticommutation relations as b and $\mathrm{b}^{\dagger}$ for $r = 1,2$.

Re-indexing the second sum of (3.14) via $\vec{p} \to -\vec{p}$ yields

$$
\Psi_L(x) = \sum_{\vec{p}} \sum_{r=1,2} \left[\mathrm{b}_{\vec{p}r} \psi^{v+}_{\vec{p}r}(x) + \mathrm{d}^{\dagger}_{\vec{p}r} \psi^{v-}_{\vec{p}r}(x) \right]
\tag{3.16}
$$

where we define

$$
\psi^{v+}_{\vec{p}r}(x) \equiv \psi^{v}_{\vec{p}r}(x) \quad \text{for} \quad r = 1,2
\tag{3.17}
$$

and

$$
\begin{aligned}
\psi^{v-}_{\vec{p}r}(x) &\equiv \sqrt{\frac{m}{VE_p}} \left[1 - \frac{e}{2p \cdot n} \gamma \cdot n \gamma \cdot A_{ext}(\eta) \right] v_{\vec{p}r} \mathrm{e}^{ip \cdot x - i \int_{-\infty}^{\eta} S^{-}(\eta') d\eta'} \\
S^{-}(\eta') &\equiv \frac{ep \cdot A_{ext}(\eta')}{p \cdot n} + \frac{e^2 A_{ext}(\eta') \cdot A_{ext}(\eta')}{2p \cdot n}
\end{aligned}
\tag{3.18}
$$

We note that (3.18) differs from (3.17) only in the substitutions $p^{\mu} \to -p^{\mu}$ and $u_{\vec{p}r} \to v_{\vec{p}r}$, now with $p^0 > 0$ for all basis functions. Eq. (3.16) is the dressed matter field operator.

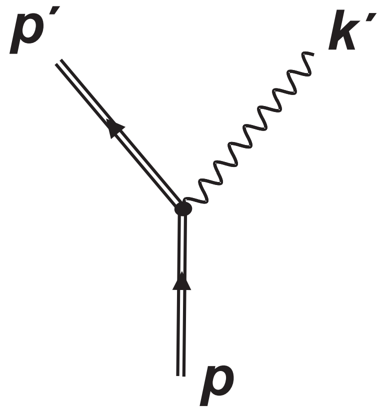


Figure 3.1 Basic vertex for Furry-Feynman diagrams.

Since the incident field has been ‘swallowed up’ by the Dirac Lagrangian, it does not appear in the quantum state of the system [37, 38]. Hence, this formulation of QED is approximate in the sense that it neglects the depletion of photons from the incident field [16]. At high intensities, we may ignore this small effect. When photon depletion cannot be ignored (ie, at lower intensities), one may construct Volkov states that non-perturbatively account for photon depletion. The Dirac equation then includes a quantized photon operator, and the Volkov states must include a ket for incident-field photons. These states are described in detail in [39]. In what follows, we assume a high-intensity incident pulse, such that photon depletion can be ignored.

3.2 Lowest-Order Scattering

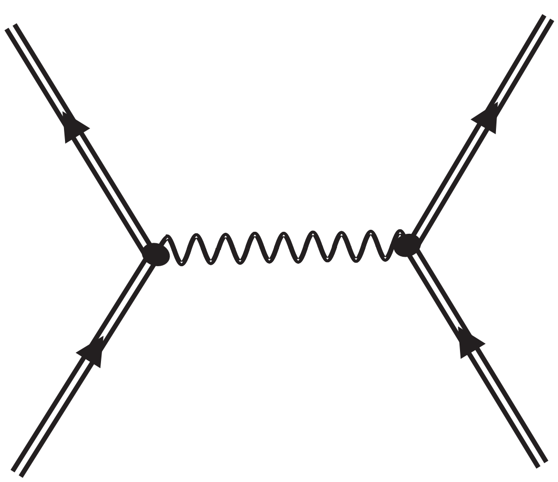
Scattering calculations in the Furry picture proceed in much the same way as in regular perturbative QED. This owes itself to the compact structure of the interaction Lagrangian density (3.1). With the classical external field separated out, the new interaction Lagrangian density is

$$\mathscr{L}_{\text{int}}(x) = -e\bar{\Psi}_L(x)\gamma_\mu\Psi_L(x)A^\mu(x) = -\mathscr{H}_{\text{int}}(x) \tag{3.19}$$

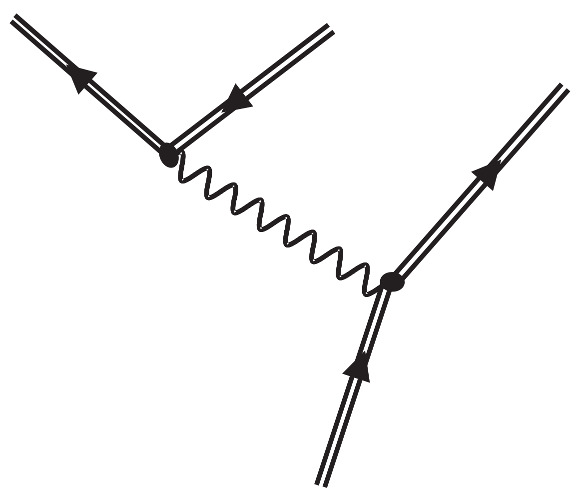
where $A^\mu(x)$ is the usual (free) photon field operator and $\mathscr{H}_{\text{int}}(x)$ is the interaction Hamiltonian density.

The fact that there are two matter operators and one photon operator in $\mathscr{H}_{\text{int}}(x)$ indicates that the basic 'Furry-Feynman' diagram has two fermion lines and one photon line. Fig. 3.1 shows the basic vertex, out of which can be constructed the diagrams for any transition. The distinguishing feature of the Furry picture is that the fermion lines are calculated from *dressed* operators; hence, it is customary to depict dressed fermion lines with double lines, or with a zig-zag pattern superimposed. Figures 3.2(a) and 3.2(b) show Furry-Feynman diagrams of laser-dressed processes recently studied in the literature, respectively electron-electron scattering [14] and trident pair production [40].

We are now prepared to compute the photoemission from a laser-dressed electron wave packet. The initial electron state is given as a superposition of Volkov states. In the context of the Furry picture, the ket $|\vec{p}\rangle$ denotes a single particle in the Volkov state parameterized by $\vec{p}$, rather than a free particle state. As in previous chapters, we will suppress spin and polarization indices. Since



(a) Laser-dressed electron-electron scattering



(b) Multi-photon trident pair production

Figure 3.2 Furry-Feynman diagrams for laser-dressed processes.

the incident light is accounted for in the dressing of the Dirac field operator, the initial quantum state contains no photons:

$$|\psi_{in}\rangle = \left(\sum_{\vec{p}} \beta_{\vec{p}} |\vec{p}\rangle\right) \otimes |0_{\vec{k}'}\rangle = \sum_{\vec{p}} \beta_{\vec{p}} |\vec{p}; 0_{\vec{k}}\rangle \tag{3.20}$$

To lowest order, we may approximate the scattering operator by the first-order term in the Dyson expansion:

$$S^{(1)} = -ie \int d^4x : \bar{\Psi}_L(x) \gamma_\mu \Psi_L(x) A^\mu(x) : \tag{3.21}$$

Only a single photon may be emitted at this order of perturbation theory, as there is only a single creation operator in $A^\mu(x)$. Hence, we project the final state $S^{(1)}|\psi_{in}\rangle$ onto basis states that contain only a single photon, as

$$\langle \vec{p}'; \vec{k}' | S^{(1)} | \psi_{in} \rangle \tag{3.22}$$

To lowest order, the square of this object equals the probability of measuring the electron and photon with (asymptotic) momenta $\vec{p}'$ and $\vec{k}'$, respectively. Fig. 3.1 shows the Furry-Feynman diagram for this transition. We note that all matrix elements of $S^{(1)}$ vanish in regular perturbative QED because of kinematic constraints. The laser dressing of the Furry picture allows for non-vanishing matrix elements of $S^{(1)}$, as the kinematics naturally include the laser photons that are excluded from the initial and final quantum states. This will become apparent in Sec. 3.3.

We first compute the electronic portion of the inner product (3.22). Inserting the fermionic operators of (3.21) into the matrix element, we find that

$$\begin{aligned}\langle \vec{p}' | : \bar{\Psi}_L(x) \gamma_\mu \Psi_L(x) : | \vec{p} \rangle &= \sum_{\vec{p}_1} \sum_{\vec{p}_2} \bar{\psi}^{v+}_{\vec{p}_1}(x) \gamma_\mu \psi^{v+}_{\vec{p}_2}(x) \langle \vec{p}' | \mathrm{b}^\dagger_{\vec{p}'_1} \mathrm{b}_{\vec{p}_2} | \vec{p} \rangle \\ &= \bar{\psi}^{v+}_{\vec{p}'}(x) \gamma_\mu \psi^{v+}_{\vec{p}}(x)\end{aligned} \tag{3.23}$$

Likewise, we find from the photon operator that

$$\begin{aligned}\langle \vec{k}' | A^\mu(x) | 0_{\vec{k}} \rangle &= \sum_{\vec{k}} \sqrt{\frac{2\pi}{kV}} \varepsilon^{\mu *}_{\vec{k}} \mathrm{e}^{ik\cdot x} \langle \vec{k}' | \mathrm{a}^\dagger_{\vec{k}} | 0_{\vec{k}} \rangle \\ &= \sqrt{\frac{2\pi}{k'V}} \varepsilon^{\mu *}_{\vec{k}'} \mathrm{e}^{ik'\cdot x}\end{aligned} \tag{3.24}$$

Inserting these results into (3.22), we find that the matrix element is

$$\langle \vec{p}\,';\vec{k}\,'|S^{(1)}|\psi_{in}\rangle = -ie\sum_{\vec{p}}\beta_{\vec{p}}\int d^4x\bar{\psi}^{\nu+}_{\vec{p}\,'}(x)\gamma_\mu\psi^{\nu+}_{\vec{p}}(x)\left(\sqrt{\frac{2\pi}{k'V}}\varepsilon^{\mu*}_{\vec{k}\,'}\mathrm{e}^{ik'\cdot x}\right) \tag{3.25}$$

It is interesting to note the resemblance that (3.25) bears to the lowest-order S-Matrix calculation from a first-quantized theory. In Appendix D, we show that this same amplitude could be computed from a semiclassical theory under the assumption that the laser-dressed Hamiltonian is perturbed by a *plane-wave* vector potential, even though the radiated field must in general be a *packet* of some sort. This supports the conclusions derived in Sec. 2.4.

3.3 Calculation of the Dressed Matrix Element

Most calculations assume that $A^\mu_{ext}(x)$ is a plane-wave field [13, 14, 41, 42]. In contrast, we consider a (unidirectional) light pulse with arbitrary spectral content. This feature has the conceptual advantage of limiting the interaction time so that the particle does not have an infinite time interval during which it can spread. Hence, the spatial size of the wave packet during the interaction is well-defined by (3.20). The consideration of arbitrary unidirectional pulses is relatively new in the literature [43–46], and our approach is unique in that overall energy-momentum conservation emerges (and generalizes) naturally.

Without loss of generality, we propagate this light in the $+z$-direction ($k_z > 0$ for all $k_z \in V_{k_z}$) and suppress the sum over polarizations:

$$A^\mu_{ext}(x) = \sum_{k_z} A_{k_z}\varepsilon^\mu_{k_z}\cos\left(k_z(t-z)+\phi_{k_z}\right) \tag{3.26}$$

where $\varepsilon^\mu_{k_z}$ represents a transverse polarization vector, and $A_{k_z} > 0$ (making the phases $\{\phi_{k_z}\}$ less ambiguous). Defining $\eta \equiv n\cdot x = t - z$, we have the unit propagation vector $n = (1,0,0,1)$.

We need energy-momentum delta functions to make kinematic arguments. To investigate this structure, we must expand the Volkov functions in (3.25) as a series of complex exponentials.

Ignoring the constant phase factor produced by the lower limit of integration in (3.4), we find that the exponent becomes:

$$\begin{aligned}\frac{1}{p\cdot n}\int^{\eta}\left(ep\cdot A_{ext}(\eta')-\frac{e^2}{2}A^2_{ext}(\eta')\right)=&\\ \frac{1}{p\cdot n}\left(e\sum_{k_z}A_{k_z}p\cdot\varepsilon_{k_z}\int^{\eta}\cos\left(k_z\eta'+\phi_{k_z}\right)d\eta'-\right.&\\ \left.\frac{e^2}{2}\sum_{k_z}\sum_{k'_z}A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}\int^{\eta}\cos\left(k_z\eta'+\phi_{k_z}\right)\cos\left(k'_z\eta'+\phi_{k'_z}\right)d\eta'\right)&\end{aligned}\tag{3.27}$$

The primed index k'_z should not be confused with the scattered photon $\vec{k}'$. Evaluating these indefinite integrals yields:

$$\begin{aligned}\frac{1}{p\cdot n}\left(e\sum_{k_z}\frac{A_{k_z}}{k_z}p\cdot\varepsilon_{k_z}\sin\left(k_z\eta+\phi_{k_z}\right)+\frac{e^2}{8}\sum_{k_z}\frac{A^2_{k_z}}{k_z}\left[2(k_z\eta+\phi_{k_z})+\sin 2(k_z\eta+\phi_{k_z})\right]\right.&\\ \left.-\frac{e^2}{4}\sum_{k_z}\sum_{k'_z\neq k_z}A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}\left[\frac{\sin\left((k_z-k'_z)\eta+\phi_{k_z}-\phi_{k'_z}\right)}{k_z-k'_z}+\frac{\sin\left((k_z+k'_z)\eta+\phi_{k_z}+\phi_{k'_z}\right)}{k_z+k'_z}\right]\right)&\end{aligned}\tag{3.28}$$

Because $\eta\equiv n\cdot x$, the middle term

$$\frac{e^2}{4p\cdot n}\sum_{k_z}A^2_{k_z}\eta \tag{3.29}$$

can be absorbed into the $p\cdot x$ term in the exponent of (3.4) to produce $q\cdot x$, where we define the *dressed* momentum 4-vector:

$$q^{\nu}\equiv p^{\nu}+\frac{e^2}{4p\cdot n}n^{\nu}\sum_{k_z}A^2_{k_z} \tag{3.30}$$

It can be shown that the dressed momentum satisfies

$$q^2=\bar{m}^2 \tag{3.31}$$

where

$$\bar{m}\equiv\sqrt{m^2+\frac{e^2}{2}\sum_{k_z}A^2_{k_z}} \tag{3.32}$$

is the dressed mass. This is the natural generalization of the case of a single-mode plane wave [13], where we now sum over k_z.

It can be algebraically shown that the integrand of (3.25) is proportional to

$$\bar{u}_{\vec{p}'}\left[1+\frac{e}{2p'\cdot n}\gamma\cdot A_{ext}(\eta)\gamma\cdot n\right]\gamma\cdot\varepsilon^*_{\vec{k}'}\left[1+\frac{e}{2p\cdot n}\gamma\cdot n\gamma\cdot A_{ext}(\eta)\right]u_{\vec{p}}\mathrm{e}^{i(q'+k'-q)\cdot x}g_1(\eta)g_2(\eta)g_3(\eta)g_4(\eta) \tag{3.33}$$

where

$$\begin{aligned}
g_1(\eta) &\equiv \exp\left[-i\sum_{k_z}\frac{e\alpha A_{k_z}}{k_z}\sin(k_z\eta+\phi_{k_z})\right]\\
g_2(\eta) &\equiv \exp\left[-i\sum_{k_z}\frac{\beta e^2A^2_{k_z}}{8k_z}\sin 2(k_z\eta+\phi_{k_z})\right]\\
g_3(\eta) &\equiv \exp\left[i\sum_{k_z}\sum_{k'_z\neq k_z}\frac{e^2\beta A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}}{4(k_z-k'_z)}\sin\Big((k_z-k'_z)\eta+\phi_{k_z}-\phi_{k'_z}\Big)\right]\\
g_4(\eta) &\equiv \exp\left[i\sum_{k_z}\sum_{k'_z\neq k_z}\frac{e^2\beta A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}}{4(k_z+k'_z)}\sin\Big((k_z+k'_z)\eta+\phi_{k_z}+\phi_{k'_z}\Big)\right]
\end{aligned} \tag{3.34}$$

and

$$\alpha\equiv\frac{p\cdot\varepsilon_{k_z}}{p\cdot n}-\frac{p'\cdot\varepsilon_{k_z}}{p'\cdot n}\quad,\quad\beta\equiv\frac{1}{p\cdot n}-\frac{1}{p'\cdot n}. \tag{3.35}$$

The $g_i(\eta)$ may be written equivalently as

$$\begin{aligned}
g_1(\eta) &= \prod_{k_z}\mathrm{e}^{-i\frac{e\alpha A_{k_z}}{k_z}\sin(k_z\eta+\phi_{k_z})}\\
g_2(\eta) &= \prod_{k_z}\mathrm{e}^{-i\frac{e^2\beta A^2_{k_z}}{8k_z}\sin 2(k_z\eta+\phi_{k_z})}\\
g_3(\eta) &= \prod_{k_z}\prod_{k'_z\neq k_z}\mathrm{e}^{i\frac{e^2\beta A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}}{4(k_z-k'_z)}\sin\left((k_z-k'_z)\eta+\phi_{k_z}-\phi_{k'_z}\right)}\\
g_4(\eta) &= \prod_{k_z}\prod_{k'_z\neq k_z}\mathrm{e}^{i\frac{e^2\beta A_{k_z}A_{k'_z}\varepsilon_{k_z}\cdot\varepsilon_{k'_z}}{4(k_z+k'_z)}\sin\left((k_z+k'_z)\eta+\phi_{k_z}+\phi_{k'_z}\right)}.
\end{aligned} \tag{3.36}$$

We may expand further using the generating function of Bessel functions [47]

$$\mathrm{e}^{iz\sin(\theta)}=\sum_{m=-\infty}^{\infty}J_m(z)\mathrm{e}^{im\theta} \tag{3.37}$$

where the $J_m(z)$ are standard Bessel functions. We find that

$$\begin{aligned}
g_1(\eta) &= \prod_{k_z}\left[\sum_{\ell} J_\ell\left(\frac{e\alpha A_{k_z}}{k_z}\right) \mathrm{e}^{-i\ell(k_z\eta+\phi_{k_z})}\right] \\
g_2(\eta) &= \prod_{k_z}\left[\sum_{m} J_m\left(\frac{e^2\beta A_{k_z}^2}{8k_z}\right) \mathrm{e}^{-i2m(k_z\eta+\phi_{k_z})}\right] \\
g_3(\eta) &= \prod_{k_z}\prod_{k_z'\neq k_z}\left[\sum_{r} J_r\left(\frac{e^2\beta A_{k_z}A_{k_z'}\varepsilon_{k_z}\cdot\varepsilon_{k_z'}}{4(k_z-k_z')}\right) \mathrm{e}^{ir\left((k_z-k_z')\eta+\phi_{k_z}-\phi_{k_z'}\right)}\right] \\
g_4(\eta) &= \prod_{k_z}\prod_{k_z'\neq k_z}\left[\sum_{s} J_s\left(\frac{e^2\beta A_{k_z}A_{k_z'}\varepsilon_{k_z}\cdot\varepsilon_{k_z'}}{4(k_z+k_z')}\right) \mathrm{e}^{is\left((k_z+k_z')\eta+\phi_{k_z}+\phi_{k_z'}\right)}\right]
\end{aligned} \tag{3.38}$$

To more easily distinguish between product expansions, we use a different summation index letter for each product expansion $g_i(\eta)$. Technically, there is a different summation index for each (infinite-sum) factor in a given product expansion, although our notation should be clear. We remark that $A^\mu_{ext}(\eta)$, as defined in (3.26), is also a sum of complex exponentials. Hence the entire integrand, as a function of x, is equivalent to products of sums of complex exponentials. We are now prepared to compute the integral over d^4x in (3.25).

The integrals over x and y are straightforward because the integrand depends on those variables only through

$$\mathrm{e}^{i(q'+k'-q)\cdot x}. \tag{3.39}$$

This indicates that (3.25) is proportional to

$$\delta\left(q'_{(x)}+k'_{(x)}-q_{(x)}\right)\delta\left(q'_{(y)}+k'_{(y)}-q_{(y)}\right) = \delta\left(p'_{(x)}+k'_{(x)}-p_{(x)}\right)\delta\left(p'_{(y)}+k'_{(y)}-p_{(y)}\right) \tag{3.40}$$

since the incident field only dresses the momentum in the direction of its propagation. (Technically, these delta functions must be of the Kronecker variety, as before, but that does not affect our conclusions.) These delta functions uniquely determine $p_{(x)}$ and $p_{(y)}$ in (3.25) in terms of $p'_{(x)}$, $p'_{(y)}$, $k'_{(x)}$, and $k'_{(y)}$– quantities that are fixed before the square is performed. That is, the sums over $p_{(x)}$ and $p_{(y)}$ collapse.

The integrals over z and t in (3.25) require more care, since $g_i(\eta)$ and $A_{ext}(\eta)$ also depend on these variables of integration. At first glance, it might appear that the sum over $p_{(z)}$ in (3.25) will not fully collapse (allowing the radiation to depend on the spatial size of the electron packet). However, integrating the sums of exponentials in (3.33) and (3.38) produces pairs of delta functions that are just right to fully collapse the sum over $p_{(z)}$, the reason being that $g_i(\eta)$ and $A_{ext}(\eta)$ depend only on z and t via exponentials of $\eta = t - z$. The important point is that the arguments of individual delta-function pairs *share* $\{k_z\}$ *dependence that can be substituted between them*. When this is done, one of the delta functions becomes identical for all pairs and can be factored out to collapse the sum over $p_{(z)}$.

To make this explicit, consider a generic exponential term of the integrand. We expand the products for each $g_i(\eta)$, enumerating k_z for $g_1(\eta)$ and $g_2(\eta)$, and enumerating pairs (k_z, k'_z) for $g_3(\eta)$ and $g_4(\eta)$. (We enumerate pairs for $g_3(\eta)$and $g_4(\eta)$ because they are double products.) Before integration, the integrand contains terms of the form

$$\begin{aligned} &\mathrm{e}^{i(q'+k'-q)\cdot x}\mathrm{e}^{-i(\ell_1 k_{z1}+\ell_2 k_{z2}+\ldots)\eta}\mathrm{e}^{-i2(m_1 k_{z1}+m_2 k_{z2}+\ldots)\eta} \\ &\times \mathrm{e}^{i\left(r_1(k_{z1}-k'_{z1})+r_2(k_{z2}-k'_{z2})+\ldots\right)\eta}\mathrm{e}^{i\left(s_1(k_{z1}+k'_{z1})+s_2(k_{z2}+k'_{z2})+\ldots\right)\eta} \end{aligned} \tag{3.41}$$

If we define

$$\begin{aligned} \Delta k_{z\{\ell_i,m_i,r_i,s_i\}} \equiv {}& (\ell_1 k_{z1}+\ell_2 k_{z2}+\ldots)+2(m_1 k_{z1}+m_2 k_{z2}+\ldots)-r_1(k_{z1}-k'_{z1}) \\ & -r_2(k_{z2}-k'_{z2})-\ldots-s_1(k_{z1}+k'_{z1})-s_2(k_{z2}+k'_{z2})-\ldots \end{aligned} \tag{3.42}$$

we find that (3.41) may be written compactly as

$$\mathrm{e}^{i(q'+k'-q)\cdot x}\mathrm{e}^{-i\Delta k_{z\{\ell_i,m_i,r_i,s_i\}}\eta} \tag{3.43}$$

When integrated over z and t, the resulting delta functions are

$$\delta\left(q'_{(0)}+k'-q_{(0)}-\Delta k_{z\{\ell_i,m_i,r_i,s_i\}}\right)\delta\left(q'_{(z)}+k'_{(z)}-q_{(z)}-\Delta k_{z\{\ell_i,m_i,r_i,s_i\}}\right) \tag{3.44}$$

We see in the kinematics that, even to lowest order, this amplitude accounts for arbitrary absorptions and re-emissions of incident-field photons [13]. This is particularly remarkable because the incident field was not quantized.

As mentioned, we can solve for $\Delta k_{z\{\ell_i,m_i,r_i,s_i\}}$ in the argument of one of the delta functions and substitute that into the other delta function. One of the delta functions becomes

$$\delta\left(q'_{(0)} - q'_{(z)} + k' - k'_{(z)} - q_{(0)} + q_{(z)}\right) \tag{3.45}$$

The definition of dressed momentum q^ν in (3.30) indicates that (3.45) is equivalent to

$$\delta\left(E_{\vec{p}'} - p'_{(z)} + k' - k'_{(z)} - E_{\vec{p}} + p_{(z)}\right) \tag{3.46}$$

which is independent of the sums over $\{\ell_i, m_i, r_i, s_i\}$. Thus, $p_{(z)}$ is uniquely determined from parameters that are fixed, and the sum over $\vec{p}$ in (3.25) is collapsed before squaring. This indicates that the relative phases of $\{\beta_{\vec{p}}\}$ do not matter, as was found for the lower-intensity case in the previous chapter. Notice that this delta function enforces a constraint that agrees with the general result (2.39) obtained in the previous chapter by use of coherent states. We note that the constraint (3.46) was also derived by alternate means in [43], where the kinematics were less transparent and harder to generalize.

This exercise also confirms the previous result that the relative phases of momenta in the incident light, here denoted by $\{\phi_{k_z}\}$, do matter, as products of sums of these phases are different for every term. We argued in Sec. 2.3 that this is expected and does not affect our conclusion that radiation scattering is independent of the electron wave-packet size.

3.4 Higher Orders of Perturbation Theory

These conclusions generalize to higher orders of perturbation theory in the Furry picture. As discussed in Sec. 3.2, all Furry-Feynman diagrams can be constructed from the basic vertex shown

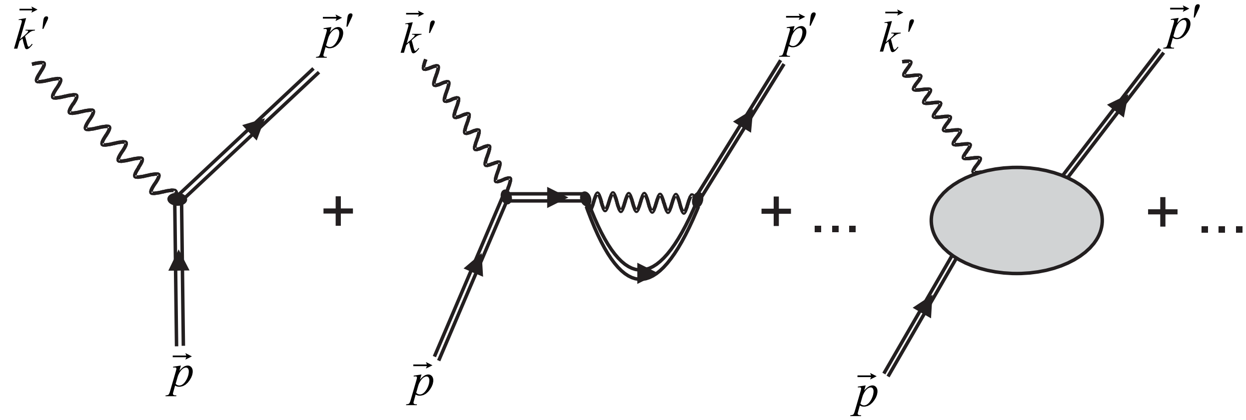


Figure 3.3 Furry-Feynman expansion for higher-order corrections to the photoemission amplitude.

in Fig. 3.1. The photon field operator is unchanged by the prescriptions of the Furry picture because only the electronic Lagrangian density was altered by (3.1) and (3.2). Hence, all external and internal photon lines are calculated in the usual way. The full amplitude

$$\langle \vec{p}\,';\vec{k}\,'|S|\vec{p};0_{\vec{k}}\rangle \tag{3.47}$$

can be computed from the Furry-Feynman diagrammatic expansion shown in Fig. 3.3. We note that the higher-order terms of (3.47) introduce only internal particle lines, as the bra and ket have only 0 and 1 for occupation numbers. This is a beneficial consequence of treating the incident field non-perturbatively.

The presence of dressed field operators in the interaction Hamiltonian density (3.19) changes the explicit calculation of internal fermion lines, but not the general structure thereof [34]. The dressed fermion propagator, a 4×4 matrix, is still computed as the time-ordered product of field operators

$$S_L\left(x,x'\right) = \langle 0|T\Psi_L(x')\bar{\Psi}_L(x)|0\rangle \tag{3.48}$$

where T is the time-ordering operator and $\Psi_L(x)$ is defined by (3.16). Inserting the expression for

$\Psi_L(x)$ yields

$$
\begin{aligned}
S_{L\alpha\beta}(x,x') &= \theta(t-t')\langle 0|\Psi_{L\alpha}(x)\bar{\Psi}_{L\beta}(x')|0\rangle - \theta(t'-t)\langle 0|\bar{\Psi}_{L\beta}(x')\Psi_{L\alpha}(x)|0\rangle \\
&= \theta(t-t')\sum_{\vec{p}r}\sum_{\vec{p}'r'}\psi^{\nu+}_{\vec{p}'r'\alpha}(x)\bar{\psi}^{\nu+}_{\vec{p}r\beta}(x')\langle 0|\mathrm{b}_{\vec{p}'r'}\mathrm{b}^{\dagger}_{\vec{p}r}|0\rangle \\
&\qquad - \theta(t'-t)\sum_{\vec{p}r}\sum_{\vec{p}'r'}\psi^{\nu-}_{\vec{p}'r'\alpha}(x)\bar{\psi}^{\nu-}_{\vec{p}r\beta}(x')\langle 0|\mathrm{d}_{\vec{p}'r'}\mathrm{d}^{\dagger}_{\vec{p}r}|0\rangle \\
&= \theta(t-t')\sum_{\vec{p}r}\psi^{\nu+}_{\vec{p}r\alpha}(x)\bar{\psi}^{\nu+}_{\vec{p}r\beta}(x') - \theta(t'-t)\sum_{\vec{p}r}\psi^{\nu-}_{\vec{p}r\alpha}(x)\bar{\psi}^{\nu-}_{\vec{p}r\beta}(x')
\end{aligned}
\tag{3.49}
$$

where we have included spinor indices α and β.

The space-time dependence of (3.49) is thus equal to a sum of products of two Volkov functions of identical parameters $\vec{p}$ and r, but different argument x. We showed in Sec. 3.3 that products of Volkov functions can be expanded as sums of complex exponentials. In this case, the generic exponential term has the form

$$
\mathrm{e}^{\pm iq\cdot(x-x')}\mathrm{e}^{i\Delta k_{z1}\eta}\mathrm{e}^{i\Delta k_{z2}\eta'} \tag{3.50}
$$

for some suitably-chosen Δk_{z1} and Δk_{z2}. When these exponentials are integrated over d^4x and d^4x' in (2.33), kinematic delta functions appear. Hence, energy-momentum is still conserved at each vertex (where the dressed momentum q^ν represents the electron), the Δk_{zi} specifying a net exchange of laser photons at each vertex. The overall energy-momentum conservation for the entire amplitude must take account of these local net exchanges with a global net exchange of laser photons. In the end, one may still define a global Δk_z that may be substituted away as described in Sec. 3.3.

The conclusion is that the sum over $\vec{p}$ in higher-order amplitudes will always collapse to the same value, dictated by the delta functions (3.40) and (3.46). These same arguments also apply to amplitudes that reflect multi-photon emission since the external lines from scattered photons enter the kinematic constraints in the usual way, as shown in (2.38). Ref. [41] computes the amplitude corresponding to Fig. 3.4, in which two photons are emitted by the laser-dressed electron. In

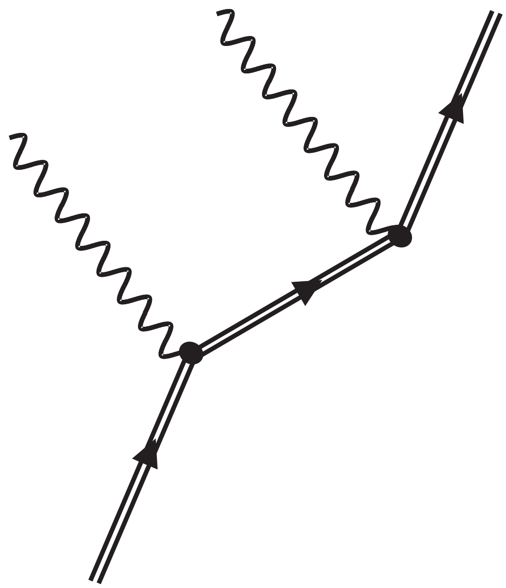


Figure 3.4 Furry-Feynman diagram for the emission of two photons.

agreement with our discussion, they find that the kinematic constraints predictably include the dressed momenta, emitted photons, and a global net exchange of laser photons.

We therefore conclude that, to all orders in a high-intensity picture, the detection of scattered photons does not depend on the size of the electron wave packet.

Chapter 4

Discussion

4.1 Unidirectionality of the Incident Pulse

In demonstrating that the probability of a scattering event is independent of the phases of $\{\beta_{\vec{p}}\}$, we used an incident pulse (3.26) traveling strictly in one direction. Since the spatial size of the initial electron wave packet can be made arbitrarily large by simply adjusting the phases via (2.41), one concludes that the strength of photon scattering is independent of the electron's wave-packet size.

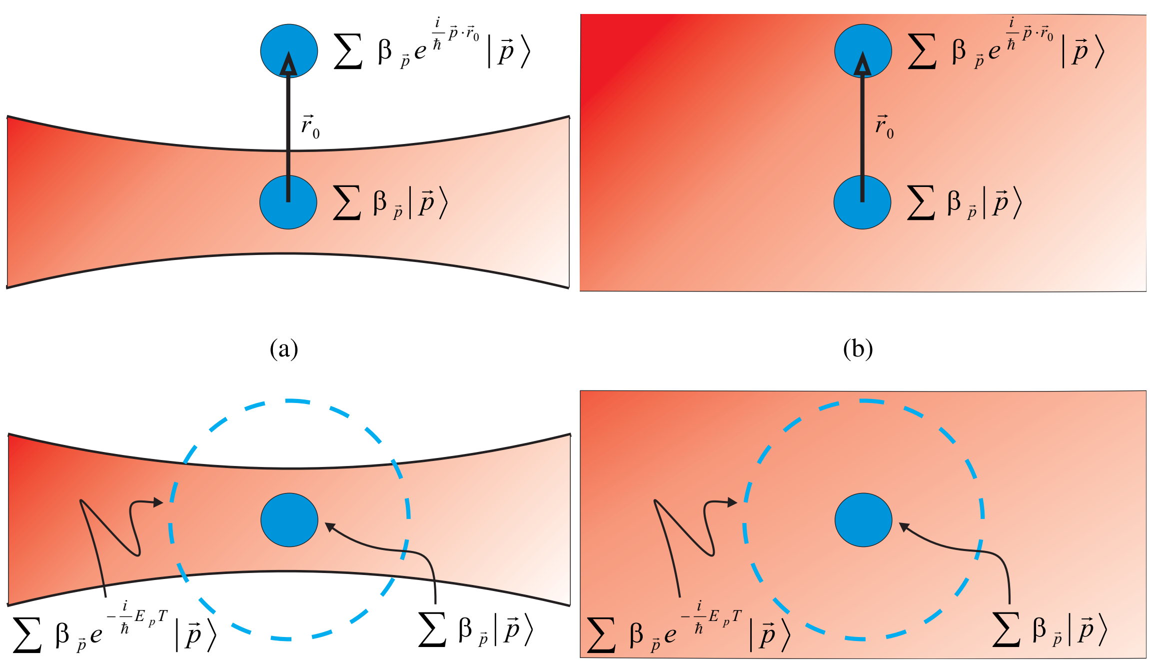
If the stimulating light is multidirectional, the scattering of the radiation does depend on the relative phases of $\{\beta_{\vec{p}}\}$. In this case, the size and shape of the electron wave packet matter. This, however, is expected and altogether ordinary. It does not negate the aforementioned conclusion.

Multidirectional light exhibits interference, which means that different regions of space can host dramatically different amounts of fluence. For example, multiple-direction modes can be used to create a focused laser beam, where a small lateral translation in position can make the difference between being inside or outside of the beam. The phases of $\{\beta_{\vec{p}}\}$ determine not only the initial size of an electron packet, but also its location, and in particular the amount of overlap with regions of high fluence. As illustrated in Fig. 4.1(a), the Fourier shift theorem can move the electron entirely out of the focus via phase adjustments. Alternatively, Fig. 4.1(c) shows that excessive (free-particle) spreading can decrease the amount of the electron wave packet that experiences the focused pulse.

It is therefore appropriate that we have addressed the radiation question under a scenario of unidirectional stimulation. It is the only way to guarantee that the entire electron wave packet (large or small) experiences the same incident light pulse, as shown in Fig. 4.1(b) and Fig. 4.1(d).

4.2 First-Quantized Matter with Quantized Light

If the incident field is strong, it may seem plausible to second-quantize only the scattered light field, keeping the electron first quantized. The electron dynamics might, in this case, satisfy the



(a) (b) (c) (d)

Figure 4.1 Comparison of momentum phase transformations for a focused light pulse and a unidirectional light pulse. Figures (a) and (b) illustrate the Fourier shift theorem, and figures (c) and (d) depict free-particle spreading.

Dirac equation dressed only by the incident field:

$$(i\gamma\cdot\partial - e\gamma\cdot A_{ext}(\eta) - m)\,\psi(x) = 0 \tag{4.1}$$

The wave function would then be represented as a superposition of Volkov states (3.4) at all times, via

$$\psi(x) = \sum_{\vec{p}r}\beta^{(0)}_{\vec{p}r}\,\psi^{v}_{\vec{p}r}(x) \tag{4.2}$$

With the electron dynamics determined by (4.2), only the photon state is changed by the interaction.

According to gauge coupling, the interaction Hamiltonian density is given by $e\,j_\mu A^\mu$, where j_μ is the Dirac probability current. Only the photon ket evolves in this picture, as radiation reaction has been ignored. That this produces a completely wrong result is evident from comparing its photoemission probability with that derived in Sec. 3.2. In the present picture, the probability of emitting a single photon $\vec{k}'$ becomes

$$\begin{aligned}\left|\langle\vec{k}'|S^{(1)}|0\rangle\right|^2 &= \left|-ie\int d^4x\bar{\psi}(x)\gamma_\mu\psi(x)\langle\vec{k}'|A^\mu(x)\,|0\rangle\right|^2\\ &= \left|-ie\sum_{\vec{p}'r'}\sum_{\vec{p}r}\beta^{(0)*}_{\vec{p}'r'}\beta^{(0)}_{\vec{p}r}\int d^4x\bar{\psi}^{v}_{\vec{p}'r'}(x)\gamma_\mu\psi^{v}_{\vec{p}r}(x)\left(\sqrt{\frac{2\pi}{k'V}}\varepsilon^{\mu*}_{\vec{k}'}\mathrm{e}^{ik'\cdot x}\right)\right|^2\end{aligned} \tag{4.3}$$

to lowest order. According to the Furry-picture calculation of Sec. 3.2, the emission probability is the square of (3.25) summed over outgoing electron states:

$$\sum_{\vec{p}'r'}\left|-ie\sum_{\vec{p}r}\beta^{(0)}_{\vec{p}}\int d^4x\bar{\psi}^{v}_{\vec{p}'r'}(x)\gamma_\mu\psi^{v}_{\vec{p}r}(x)\left(\sqrt{\frac{2\pi}{k'V}}\varepsilon^{\mu*}_{\vec{k}'}\mathrm{e}^{ik'\cdot x}\right)\right|^2 \tag{4.4}$$

These quantities are manifestly unequal, as (4.3) contains an extra sum $\sum_{\vec{p}'r'}\beta^{(0)*}_{\vec{p}'r'}$ inside of the square. Thus, neglecting the electronic state as a dynamical variable leads (via standard gauge coupling) to incorrect emission probabilities.

We now show explicitly that (4.3) incorrectly gives rise to radiative interference, even though the light field is second-quantized. For simplicity, we consider the case in which detected photons are linearly polarized (such that $\varepsilon^*_{\vec{k}'} = \varepsilon_{\vec{k}'}$). Since the radiated photon is transverse, the lowest-order scattering amplitude is proportional to

$$\int d^4x \vec{J}(\vec{x},t) \cdot \hat{\varepsilon}_{\vec{k}'} \mathrm{e}^{i\left(k't - \vec{k}' \cdot \vec{x}\right)} \tag{4.5}$$

where $\vec{J}(\vec{x},t)$ is the probability current. Suppose that $\hat{\varepsilon}_{\vec{k}'}$ and $\hat{z}$ are orthogonal and define a plane wherein $\vec{J}(\vec{x},t)$ oscillates. (This is depicted in Fig. 1.1(b), where $\hat{z}$ and $\hat{\varepsilon}_{\vec{k}'}$ respectively orient the horizontal and vertical axes.) Suppose also that $\vec{J}(\vec{x},t) \cdot \hat{\varepsilon}_{\vec{k}'}$ is approximately an odd function of z. It then follows, from parity in z, that (4.5) approximately vanishes for photons radiated in the direction $\hat{k}' = \hat{\varepsilon}_{\vec{k}'} \times \hat{z}$. This destructive interference along $\hat{k}'$ is a consequence of treating the probability current as a *classical* charge current in the interaction Hamiltonian. As we have shown, this contradicts the QED prediction for single-electron photoemission.

4.3 Summary

We have shown that the spatial size of a laser-driven electron wave packet has no effect on photoemission if the stimulating light is unidirectional. Using coherent states of light and the scattering theory of quantum electrodynamics, we showed that energy-momentum conservation forbids interference in the scattered light at every order of perturbation theory. A crucial premise of this analysis is the Born rule that probabilities are computed by projecting the state vector onto a basis eigenvector, squaring the projection amplitude, and *then* summing over a set of basis eigenvalues.

Working in the Furry picture of QED, we considered the possibility that interference may arise in the high-intensity limit. Quantizing the matter field with Volkov functions treats the incident field non-perturbatively. We found that a similar kinematic structure emerges – forbidding radiative interference at every order of perturbation theory.

We have also connected to models where the electron remains first-quantized. Importantly, we found that it is not appropriate to generate the scattered radiation field from the probability current.

We also showed how to match first-quantized scattering amplitudes to those predicted by lowest-order QED. This prescription chooses the scattered radiation field to be a single mode of energy $\hbar ck'$. One must then reinterpret the transition amplitude to be multi-particle in nature.

Classical electrodynamics dictates that emissions from different regions of a charge current add coherently. This is clearly not true for probability currents. The subtleties of quantum electrodynamics require a new intuition.

Appendix A

Klein-Gordon Perturbation Theory

In this appendix, we derive the lowest-order scattering amplitude for a first-quantized Klein-Gordon particle. For the sake of completeness, we include the possibility that the incident field contains a zeroth component A_i^0. The virtue of doing this is that it produces a manifestly Lorentz-invariant scattering amplitude.

We begin by substituting the perturbative expansion (1.12) into the perturbed wave equation (1.10). As the $\{\lambda^m\}$ are all linearly independent, we may set the coefficient of each λ^m equal to zero. The coefficient of λ^0 vanishes identically, as it is equivalent to the unperturbed problem. For the λ^1 term, we find that

$$0 = \sum_{\vec{p}} \left\{ \left[\left(i\hbar\partial^\mu - \frac{e}{c} A_i^\mu \right)^2 - m^2c^2 \right] \beta_{\vec{p}}^{(1)}(t)\, \psi_{\vec{p}}^{\mathrm{v}} + V_{\mathrm{int}} \beta_{\vec{p}}^{(0)} \psi_{\vec{p}}^{\mathrm{v}} \right\} \tag{A.1}$$

A simple calculation shows that the commutator

$$\left[i\hbar\partial^\mu - \frac{e}{c} A_i^\mu, f(t) \right] = i\hbar \delta_0{}^\mu \left(\partial^0 f \right) \tag{A.2}$$

holds for any differentiable function $f(t)$. We will use this commutator to move $\beta_{\vec{p}}^{(1)}(t)$ leftwards in (A.1). Commuting once yields

$$0=\sum_{\vec{p}}\left\{\left(i\hbar\partial^{\mu}-\frac{e}{c}A_i^{\mu}\right)\left[ih\delta^0{}_{\mu}\left(\partial^0\beta_{\vec{p}}^{(1)}\right)+\beta_{\vec{p}}^{(1)}\left(ih\partial_{\mu}-\frac{e}{c}A_{i\mu}\right)\right]\psi_{\vec{p}}^{\mathrm{v}}-m^2c^2\beta_{\vec{p}}^{(1)}\psi_{\vec{p}}^{\mathrm{v}}+V_{\mathrm{int}}\beta_{\vec{p}}^{(0)}\psi_{\vec{p}}^{\mathrm{v}}\right\} \tag{A.3}$$

and commuting a second time yields

$$\begin{aligned}0=\sum_{\vec{p}}\Big\{\Big[&(i\hbar)^2\delta^0{}_{\mu}\delta_0{}^{\mu}\left(\partial^0\partial^0\beta_{\vec{p}}^{(1)}\right)+i\hbar\delta^0{}_{\mu}\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\left(i\hbar\partial^{\mu}-\frac{e}{c}A_i^{\mu}\right)\\&+i\hbar\delta_0{}^{\mu}\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\left(i\hbar\partial_{\mu}-\frac{e}{c}A_{i\mu}\right)+\beta_{\vec{p}}^{(1)}\left(i\hbar\partial^{\mu}-\frac{e}{c}A_i^{\mu}\right)^2\Big]\psi_{\vec{p}}^{\mathrm{v}}\\&-m^2c^2\beta_{\vec{p}}^{(1)}\psi_{\vec{p}}^{\mathrm{v}}+V_{\mathrm{int}}\beta_{\vec{p}}^{(0)}\psi_{\vec{p}}^{\mathrm{v}}\Big\}\end{aligned} \tag{A.4}$$

Canceling terms from the unperturbed problem, we find that

$$0=\sum_{\vec{p}}\left\{\left[(i\hbar)^2\left(\partial^0\partial^0\beta_{\vec{p}}^{(1)}\right)+2i\hbar\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\left(i\hbar\partial^0-\frac{e}{c}A_i^0\right)\right]\psi_{\vec{p}}^{\mathrm{v}}+V_{\mathrm{int}}\psi_{\vec{p}}^{\mathrm{v}}\right\} \tag{A.5}$$

We now multiply on the left by $\psi_{\vec{p}'}^{\mathrm{v}*}$ and integrate over d^4x. The term with $\left(\partial^0\partial^0\beta_{\vec{p}}^{(1)}\right)$ can be integrated by parts over dx^0. We see that

$$\begin{aligned}\int dx^0\left(\partial^0\partial^0\beta_{\vec{p}}^{(1)}\right)\psi_{\vec{p}'}^{\mathrm{v}*}\psi_{\vec{p}}^{\mathrm{v}}&=\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\psi_{\vec{p}'}^{\mathrm{v}*}\psi_{\vec{p}}^{\mathrm{v}}\Big|_{-\infty}^{+\infty}-\int dx^0\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\partial^0\left(\psi_{\vec{p}'}^{\mathrm{v}*}\psi_{\vec{p}}^{\mathrm{v}}\right)\\&=-\int dx^0\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\left[\left(\partial^0\psi_{\vec{p}'}^{\mathrm{v}*}\right)\psi_{\vec{p}}^{\mathrm{v}}+\psi_{\vec{p}'}^{\mathrm{v}*}\left(\partial^0\psi_{\vec{p}}^{\mathrm{v}}\right)\right]\end{aligned} \tag{A.6}$$

where the boundary terms vanish if the incident pulse is of finite duration. We find that

$$0=\sum_{\vec{p}}\left\{(i\hbar)^2\int d^4x\left(\partial^0\beta_{\vec{p}}^{(1)}\right)\left(\psi_{\vec{p}'}^{\mathrm{v}*}\partial^0\psi_{\vec{p}}^{\mathrm{v}}-\psi_{\vec{p}}^{\mathrm{v}}\partial^0\psi_{\vec{p}'}^{\mathrm{v}*}-\frac{2e}{i\hbar c}A_i^0\psi_{\vec{p}'}^{\mathrm{v}*}\psi_{\vec{p}}^{\mathrm{v}}\right)+\int d^4x\psi_{\vec{p}'}^{\mathrm{v}*}V_{\mathrm{int}}\psi_{\vec{p}}^{\mathrm{v}}\right\} \tag{A.7}$$

We can compute the integral over d^3x on the left, as it is the orthonormality integral in (1.9). The result is

$$0=\int dx^0\left(\partial^0\beta_{\vec{p}'}^{(1)}\right)(2i\hbar mc)+\sum_{\vec{p}}\int d^4x\psi_{\vec{p}'}^{\mathrm{v}*}V_{\mathrm{int}}\psi_{\vec{p}}^{\mathrm{v}} \tag{A.8}$$

Using the fundamental theorem of calculus and the initial condition that $\beta_{\vec{p}'}^{(1)}(-\infty)=0$, we can solve for the transition amplitude:

$$\beta_{\vec{p}'}^{(1)}(\infty)=\frac{i}{2\hbar mc}\sum_{\vec{p}}\beta_{\vec{p}}^{(0)}\int d^4x\psi_{\vec{p}'}^{\mathrm{v}*}V_{\mathrm{int}}\psi_{\vec{p}}^{\mathrm{v}} \tag{A.9}$$

Appendix B

Review of Quantized Field Operators

For the reader's convenience, we now review the free quantum field operators that arise in QED. We follow convention by scaling units such that $\hbar$ and c vanish from the expressions. Our electromagnetic units are not rationalized, such that we retain the factors of 4π common to Gaussian units.

The Dirac field operator may be expanded in free-particle wave functions:

$$\Psi(x) = \sum_{\vec{p}r} \sqrt{\frac{m}{E_p V}} \left[\mathrm{b}_{\vec{p}r} u_{\vec{p}r} \mathrm{e}^{-ip\cdot x} + \mathrm{d}^{\dagger}_{\vec{p}r} v_{\vec{p}r} \mathrm{e}^{ip\cdot x} \right] \tag{B.1}$$

where $u_{\vec{p}r}$ and $v_{\vec{p}r}$ are Dirac spinors. The anticommutators (1.27) and (1.28) can be used to show that

$$\begin{aligned} \left\{ \mathrm{b}_{\vec{p}r}, \mathrm{b}^{\dagger}_{\vec{p}'r'} \right\} = \left\{ \mathrm{d}_{\vec{p}r}, \mathrm{d}^{\dagger}_{\vec{p}'r'} \right\} &= \delta_{\vec{p}\vec{p}'} \delta_{rr'} \\ \left\{ \mathrm{b}_{\vec{p}r}, \mathrm{b}_{\vec{p}'r'} \right\} = \left\{ \mathrm{b}^{\dagger}_{\vec{p}r}, \mathrm{b}^{\dagger}_{\vec{p}'r'} \right\} = \left\{ \mathrm{d}_{\vec{p}r}, \mathrm{d}_{\vec{p}'r'} \right\} &= \left\{ \mathrm{d}^{\dagger}_{\vec{p}r}, \mathrm{d}^{\dagger}_{\vec{p}'r'} \right\} = 0 \end{aligned} \tag{B.2}$$

We interpret the b and d operators as annihilators of electrons and positrons, respectively; likewise, their adjoints create electrons and positrons.

When quantized in the Lorenz gauge, the photon field operator can be expanded in plane waves as

$$A^{\mu}(x) = \sum_{\vec{k}\lambda} \sqrt{\frac{2\pi}{Vk}} \left[a_{\vec{k}\lambda} \varepsilon^{\mu}_{\vec{k}\lambda} \mathrm{e}^{-ik\cdot x} + a^{\dagger}_{\vec{k}\lambda} \varepsilon^{\mu *}_{\vec{k}\lambda} \mathrm{e}^{ik\cdot x} \right] \tag{B.3}$$

The creation/annihilation operators satisfy the following commutation relation:

$$\left[a_{\vec{k}\lambda}, a^{\dagger}_{\vec{k}'\lambda'}\right] = -\delta_{\vec{k}\vec{k}'} g^{\lambda\lambda'} \tag{B.4}$$

This commutator is physically problematic because it allows for the existence of (unmeasurable) scalar and longitudinal photon states. Moreover, the scalar photon states have *negative* norm. From a calculational standpoint, it is customary to skirt these issues by considering only transverse photons in the initial and final state of the system [21]. One may alternatively quantize in the Coulomb gauge, but the expressions for the interaction Hamiltonian and photon propagator become unwieldy and are not manifestly covariant. It can be shown that both quantization schemes yield identical results for measurable transition probabilities. Ref. [34] provides an overview of various treatments of the quantized light field.

The creation operators $a^{\dagger}_{\vec{k}\lambda}$, $\mathrm{b}^{\dagger}_{\vec{p}r}$, and $\mathrm{d}^{\dagger}_{\vec{p}r}$ may be used to construct the number states of QED. These operators increase the occupation number of a given mode by 1. For bosons, we have that

$$a^{\dagger}_{\vec{k}\lambda} \left|\ldots n_{\vec{k}\lambda} \ldots\right\rangle = \left(n_{\vec{k}\lambda} + 1\right)^{1/2} \left|\ldots, n_{\vec{k}\lambda} + 1, \ldots\right\rangle \tag{B.5}$$

A photon state with arbitrary occupation numbers can be constructed from the vacuum via

$$\left|\{n_{\vec{k}\lambda}\}\right\rangle = \left(\prod_{\vec{k}\lambda} \frac{\left(a^{\dagger}_{\vec{k}\lambda}\right)^{n_{\vec{k}\lambda}}}{\sqrt{n_{\vec{k}\lambda}!}}\right) |0\rangle \tag{B.6}$$

The orthonormality relation for these states is

$$\left\langle \{m_{\vec{k}\lambda}\} \middle| \{n_{\vec{k}\lambda}\} \right\rangle = \prod_{\vec{k}\lambda} \delta_{n_{\vec{k}\lambda} m_{\vec{k}\lambda}} \tag{B.7}$$

The fermion creation operators likewise generate number states, although the maximum occupation number for a given mode is 1. This feature is contained naturally in the fact that all $b^{\dagger}$'s and $\mathrm{d}^{\dagger}$'s anticommute with themselves:

$$\mathrm{b}^{\dagger}_{\vec{p}r} \mathrm{b}^{\dagger}_{\vec{p}r} |0\rangle = -\mathrm{b}^{\dagger}_{\vec{p}r} \mathrm{b}^{\dagger}_{\vec{p}r} |0\rangle = 0 \tag{B.8}$$

It may be shown similarly that general fermionic (bosonic) states are antisymmetric (symmetric) with respect to exchange of particles.

The annihilation operators lower the occupation number of a given mode by 1, producing the zero-vector if the mode is already unoccupied. For bosons, we have

$$\begin{aligned} a_{\vec{k}\lambda} \left|\ldots, n_{\vec{k}\lambda}, \ldots\right\rangle &= n_{\vec{k}\lambda}^{1/2} \left|\ldots, n_{\vec{k}\lambda} - 1, \ldots\right\rangle \\ a_{\vec{k}\lambda} \left|\ldots, 0_{\vec{k}\lambda} \ldots\right\rangle &= 0 \end{aligned} \tag{B.9}$$

with a similar relation holding for b and d. Combining (B.5) with (B.9), we can construct the single-mode photon number operator

$$a_{\vec{k}\lambda}^{\dagger} a_{\vec{k}\lambda} \left|\ldots, n_{\vec{k}\lambda}, \ldots\right\rangle = n_{\vec{k}\lambda} \left|\ldots, n_{\vec{k}\lambda}, \ldots\right\rangle \tag{B.10}$$

We make use of this operator in Sec. 2.2.

Appendix C

Volkov Functions as Solutions to the Dirac Equation

Here we show that the Dirac Volkov functions satisfy the Dirac equation

$$(i\gamma \cdot \partial - e\gamma \cdot A - m)\, \psi^{\nu}_{\vec{p}r} = 0 \tag{C.1}$$

These functions are given by (3.4) and (3.5):

$$\psi^{\nu}_{\vec{p}r}(x) = \sqrt{\frac{m}{V|E_p|}} \left[1 + \frac{e}{2p \cdot n}\gamma \cdot n\gamma \cdot A(\eta)\right] e^{-ip\cdot x - i\int_{-\infty}^{\eta} S(\eta')d\eta'} u_{\vec{p}r} \tag{C.2}$$

where

$$S(\eta') = \frac{ep \cdot A_{(}\eta')}{p \cdot n} - \frac{e^2 A(\eta') \cdot A(\eta')}{2p \cdot n} \tag{C.3}$$

We will need to make use of the Dirac anticommutator [19], given by

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \tag{C.4}$$

Multiplying both sides of (C.4) by $a_{\mu}b_{\nu}$ yields the identity

$$\not{a}\ \not{b} + \not{b}\ \not{a} = 2a \cdot b \tag{C.5}$$

where we define $\not{a} \equiv \gamma \cdot a$. In the event that $a = b$, we have $\not{a}\ \not{a} = a \cdot a$.

We first examine the operation of the $\gamma \cdot \partial$ term of (C.1) on a Volkov function. Operating first on the non-exponential factor of (C.2), we see that

$$\begin{aligned} \gamma \cdot \partial \left[1 + \frac{e}{2p \cdot n} \gamma \cdot n \gamma \cdot A(\eta)\right] u_{\vec{p}r} &= \gamma \cdot (\partial \eta) \frac{e}{2p \cdot n} \gamma \cdot n \gamma \cdot (\frac{dA}{d\eta}) u_{\vec{p}r} \\ &= \frac{e}{2p \cdot n} \gamma \cdot n \gamma \cdot n \gamma \cdot (\frac{dA}{d\eta}) u_{\vec{p}r} \\ &= 0 \end{aligned} \tag{C.6}$$

since $\gamma \cdot n \gamma \cdot n = n \cdot n = 0$. Using the chain rule, we find that

$$\begin{aligned} i\gamma_\mu \partial^\mu \psi^\nu_{\vec{p}r} =& \gamma_\mu \left(p^\mu + (\partial^\mu \eta) S(\eta)\right) \psi^\nu_{\vec{p}r} \\ =& \left(\not{p} + \not{n} S(\eta)\right) \psi^\nu_{\vec{p}r} \end{aligned} \tag{C.7}$$

Hence, we can write (C.1) as

$$\left(\not{p} + \not{n} S(\eta) - e \not{A}(\eta) - m\right) \psi^\nu_{\vec{p}r} = 0 \tag{C.8}$$

If this equality holds, we may divide out the exponential factor of (C.2) and write

$$\left(\not{p} + \not{n} S(\eta) - e \not{A}(\eta) - m\right) \left[1 + \frac{e}{2p \cdot n} \not{n} \not{A}(\eta)\right] u_{\vec{p}r} = 0 \tag{C.9}$$

We expand the left-hand side (noting again that $\not{n}\ \not{n} = n \cdot n = 0$) as

$$\left[\not{p} + \frac{e}{2p \cdot n}(2p \cdot n \not{A} - \not{n} \not{p} \not{A}) + S(\eta) \not{n} - e \not{A} - \frac{e^2}{2p \cdot n} \not{A} \not{n} \not{A} - m\left(1 + \frac{e}{2p \cdot n} \not{n} \not{A}\right)\right] u_{\vec{p}r} \tag{C.10}$$

where we have used the identity (C.5) to produce the second term above. We see right away that the $e \not{A}$ terms cancel. Using (C.5) again on the $\not{n}\ \not{p}\ \not{A}$ and $\not{A}\ \not{n}\ \not{A}$ terms yields

$$\begin{aligned} \Bigg[\not{p} - \frac{e}{p \cdot n} p \cdot A \not{n} + \frac{e}{2p \cdot n} \not{n} \not{A} \not{p} + S(\eta) \not{n} - \frac{e^2}{p \cdot n} n \cdot A \not{A} \\ + \frac{e^2}{2p \cdot n} \not{A} \not{A} \not{n} - m\left(1 + \frac{e}{2p \cdot n} \not{n} \not{A}\right)\Bigg] u_{\vec{p}r} \end{aligned} \tag{C.11}$$

From the definition of $S(\eta)$ given in (C.3) and the fact that $\not{A}\not{A} = A \cdot A$, we see that the $S(\eta)\not{n}$ terms cancel. Hence, (C.9) is equivalent to

$$\left[\left(1+\frac{e}{2p\cdot n}\not{n}\not{A}\right)(\not{p}-m)-\frac{e^2}{p\cdot n}n\cdot A\not{A}\right]u_{\vec{p}r}=0 \tag{C.12}$$

Evidently, the Volkov functions satisfy the Dirac equation if

$$\begin{aligned} n\cdot A(\eta) &= 0 \\ (\not{p}-m)u_{\vec{p}r} &= 0 \end{aligned} \tag{C.13}$$

The first condition constrains the incident field to satisfy a covariant version of transversality. The second condition is the defining relation for the free-particle Dirac spinor $u_{\vec{p}r}$.

Appendix D

Dirac Perturbation Theory

We now derive the lowest-order scattering amplitude for a first-quantized Dirac particle. We begin with the Dirac equation

$$(i\gamma\cdot\partial - e\gamma\cdot A_{ext}(x) - m)\,\psi(x) - \lambda e\gamma\cdot A_s(x)\psi(x) = 0 \tag{D.1}$$

where we have separated the external field A_{ext} from the scattered field A_s. The perturbation parameter λ identifies the 'small' quantity $eA_s(x)$. The unperturbed problem ($\lambda = 0$) is solved by a superposition of Volkov functions (see Sec 3.1):

$$\psi(x) = \sum_{\vec{p}r} \beta^{(0)}_{\vec{p}r}\,\psi^{v}_{\vec{p}r}(x) \tag{D.2}$$

where $\beta^{(0)}_{\vec{p}r}$ is independent of time. The solution to the perturbed problem may also be expanded in the basis of Volkov functions, although the coefficients of expansion must now be time-dependent:

$$\psi(x) = \sum_{\vec{p}r} \left(\beta^{(0)}_{\vec{p}r} + \lambda\beta^{(1)}_{\vec{p}r}(t) + \ldots\right)\psi^{v}_{\vec{p}r}(x) \tag{D.3}$$

We take the boundary condition that $\beta^{(i)}_{\vec{p}r}(-\infty) = 0$ for $i \geq 1$. Only lowest-order terms will be considered in this calculation.

Substituting the ansatz (D.3) into (D.1) produces:

$$\sum_{\vec{p}r}\Big[i\gamma\cdot\Big(\partial\lambda\beta^{(1)}_{\vec{p}r}(t)\ +\dots\Big)\,\psi^{v}_{\vec{p}r}(x)$$
$$+\Big(\beta^{(0)}_{\vec{p}r}+\lambda\beta^{(1)}_{\vec{p}r}(t)+\dots\Big)\,(i\gamma\cdot\partial-e\gamma\cdot A_{ext}-\lambda e\gamma\cdot A_s-m)\,\psi^{v}_{\vec{p}r}(x)\Big]=0 \tag{D.4}$$

Terms proportional to λ^0 neatly vanish from the expression, as the Volkov functions satisfy the Dirac equation dressed only by the incident field A_{ext}. We take advantage of the linear independence of $\{\lambda^m\}$ by setting their respective coefficients equal to zero. For λ^1, we find that

$$\sum_{\vec{p}r}\Big[i\gamma^0\dot{\beta}^{(1)}_{\vec{p}r}(t)\,\psi^{v}_{\vec{p}r}(x)-e\beta^{(0)}_{\vec{p}r}\,\gamma\cdot A_s(x)\,\psi^{v}_{\vec{p}r}(x)\Big]=0 \tag{D.5}$$

Multiplying on the right by $\bar{\psi}^{v}_{\vec{p}'r'}(x)$, and noting that $\bar{\psi}\gamma^0=\psi^\dagger$, we find that

$$\sum_{\vec{p}r}\Big[i\dot{\beta}^{(1)}_{\vec{p}r}(t)\,\psi^{v\dagger}_{\vec{p}'r'}(x)\,\psi^{v}_{\vec{p}r}(x)-e\beta^{(0)}_{\vec{p}r}\,\bar{\psi}^{v}_{\vec{p}'r'}(x)\,\gamma\cdot A_s(x)\,\psi^{v}_{\vec{p}r}(x)\Big]=0 \tag{D.6}$$

If we integrate both sides over volume V, we can exploit the orthonormality of the Volkov functions, finding that

$$\dot{\beta}^{(1)}_{\vec{p}'r'}(t)=-ie\sum_{\vec{p}r}\beta^{(0)}_{\vec{p}r}\int d^3x\,\bar{\psi}^{v}_{\vec{p}'r'}(x)\,\gamma\cdot A_s(x)\,\psi^{v}_{\vec{p}r}(x) \tag{D.7}$$

Last of all, we integrate both sides over $t\in(-\infty,\infty)$ to find the scattering amplitude

$$\beta^{(1)}_{\vec{p}'r'}(\infty)=-ie\sum_{\vec{p}r}\beta^{(0)}_{\vec{p}r}\int d^4x\,\bar{\psi}^{v}_{\vec{p}'r'}(x)\,\gamma\cdot A_s(x)\,\psi^{v}_{\vec{p}r}(x) \tag{D.8}$$

where we've imposed the boundary condition that $\beta^{(1)}_{\vec{p}'r'}(-\infty)=0$. For intensities where pair creation can be ignored (as must be the case for an intrinsically *single*-particle theory), the post-interaction state must be a single particle of positive energy ($r=1,2$). At higher orders of perturbation theory, one must include intermediate states of both positive and negative energies [16].

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